On the Global Minimizers of a Nonlocal Isoperimetric Problem in Two Dimensions

Peter Sternberg* Ihsan Topaloglu
sternber@indiana.edu itopalog@indiana.edu
Department of Mathematics, Indiana University, Bloomington, IN

October 7, 2010

Abstract

In this article we analyze the minimization of a nonlocal isoperimetric problem (NLIP) posed on the flat 2-torus. After establishing regularity of the free boundary of minimizers, we show that when the parameter controlling the influence of the nonlocality is small, there is an interval of values for the mass constraint such that the global minimizer is exactly lamellar; that is, the free boundary consists of two parallel lines. In other words, in this parameter regime, the global minimizer of the 2d (NLIP) coincides with the global minimizer of the local periodic isoperimetric problem.

1 Introduction

The nonlocal isoperimetric problem (NLIP) is given by

$$\text{minimize} \quad E_\gamma(u) := \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u| + \gamma \int_{\mathbb{T}^2} |\nabla v|^2\,dx, \tag{1.1}$$

over all $u \in BV(\mathbb{T}^2, \{\pm 1\})$ satisfying

$$\int_{\mathbb{T}^2} u\,dx = m$$

and v satisfying

$$-\Delta v = u - m \;\text{ in } \mathbb{T}^2 \quad \text{with} \quad \int_{\mathbb{T}^2} v\,dx = 0. \tag{1.2}$$

Here $\mathbb{T}^2$ is the flat 2-torus and the first term in E_γ computes the perimeter of the set $\{x : u(x) = 1\}$. For an interval of m-values containing $m = 0$ and for γ small, we will show that the global minimizer is lamellar; that is, the set $\{x : u(x) = 1\}$ is simply a strip.

*corresponding author

The problem (NLIP) arises, up to a constant factor, as the Γ-limit as $\varepsilon \to 0$ of the well-studied Ohta-Kawasaki sequence of functionals $E_{\varepsilon,\gamma}$ which model microphase separation of diblock copolymers, [1, 18]:

$$E_{\varepsilon,\gamma}(u) := \begin{cases} \int_{\mathbb{T}^2} \frac{\varepsilon}{2}|\nabla u|^2 + \frac{(1-u^2)^2}{4\varepsilon} + \gamma\,|\nabla v|^2\,dx & \text{if } u \in H^1(\mathbb{T}^2) \\ & \text{ and } \int_{\mathbb{T}^2} u\,dx = m, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.3}$$

where again v satisfies (1.2). There is an extensive literature exploring the energy landscape for $E_{\varepsilon,\gamma}$ in two and three dimensions, whether posed on the flat torus (i.e. with periodic boundary conditions) or on a general domain with homogeneous Neumann data, cf. e.g. [3, 20, 21, 22, 23, 24]. The picture is quite rich and complicated, with the diffuse interface sometimes bounding one or more strips, wriggled strips, discs or ovals.

Much the same richness exists for the energy landscape of (NLIP). As such, independent of its connection to Ohta-Kawasaki, (NLIP) attracts interest as a rather canonical nonlocal perturbation of the classical isoperimetric problem. Indeed, as a model for pattern formation, (NLIP) sets up a basic competition between low surface area (the perimeter term) and high oscillation (the nonlocal term).

In three dimensions, computations reveal a wide array of stable critical points, with the free boundary $\partial\{x : u(x) = 1\}$ consisting of one or more pairs of parallel planes, one or more spheres, cylinders or even hypersurfaces resembling more exotic triply periodic constant mean curvature surfaces such as gyroids, depending on where in the (m, γ) parameter space one looks. With few exceptions, however, rigorous proofs of stability for particular patterns are rare (cf. e.g. [2, 25, 26, 27]), and to our knowledge, there are no proofs of global or even local minimality of specific critical points. In this regard, we mention the interesting investigation of [19], in which the authors seek to show that a lamellar (striped) pattern minimizes energy for a slightly different model related to diblock copolymers. Commenting on the inherent difficulty in picking out such a pattern as the "winner" in an energy landscape full of locally minimizing competitors, the authors of [19] remark, "...when comparing a striped pattern with arbitrary multidimensional patterns we know of no rigorous results, for any system." We also note the recent work [17] on a characterization of minimizers in a related model including screened Coulomb interaction in the setting of small volume fraction. There the author shows that minimizers form a collection of nearly identical circular droplets.

Here we have chosen to focus on the two-dimensional setting of (NLIP) with γ small in order to present what is perhaps the first rigorous proof that a particular pattern is globally minimizing. To be more specific, fixing any $m \in (-1, 1)$ let us define the lamellar function $u_L : \mathbb{T}^2 \to \mathbb{R}$ whose phase $\{x : u(x) = 1\}$ occupies in $[0, 1] \times [0, 1]$ a single vertical strip given by

$$u_L(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in (\frac{1}{4} - \frac{m}{2}, \frac{3}{4}), \\ -1 & \text{if } x_1 \notin (\frac{1}{4} - \frac{m}{2}, \frac{3}{4}). \end{cases} \tag{1.4}$$

One easily checks that u_L is a critical point of E_γ for all γ. (See (2.3) for the precise characterization of criticality.) Furthermore, for small γ it is stable in the sense of nonnegative second variation, as shown in [5, Proposition 3.5], and it is unstable for larger γ, cf. [5, Proposition 3.6], and [22]. However, in this article we go further to establish the *global minimality* of this lamellar critical point for small γ when m lies in the interval $(\frac{2}{\pi}-1, 1-\frac{2}{\pi})$. For $\gamma = 0$, we note that the problem reduces to the well-known (local) periodic isoperimetric problem (cf. [4, 10]):

$$\begin{gathered} \text{minimize} \quad E_0(u) := \tfrac{1}{2}\int_{\mathbb{T}^2} |\nabla u| \\ \text{over all } u \in BV(\mathbb{T}^2, \{\pm 1\}) \text{ subject to } \int_{\mathbb{T}^2} u\, dx = m. \end{gathered} \tag{1.5}$$

For this problem the lamellar critical point u_L is known to be the global minimizer for $m \in (\frac{2}{\pi}-1, 1-\frac{2}{\pi})$ in two dimensions, cf. [12]. Thus, the restriction on the mass m in our result is inherited from the global minimality of the strip in the local periodic isoperimetric problem, and is surely a necessary condition for global minimality with respect to E_γ for $\gamma > 0$. In light of the Γ-convergence of E_γ to E_0 it is easy to see that the minimizer of E_γ, say u_γ, is close to u_L when γ is small, as shown in Proposition 3.1. We emphasize, however, that our main result, Theorem 3.3, says more, that is, $u_\gamma \equiv u_L$.

Our argument requires the regularity of minimizers to (NLIP). This regularity will be established in the next section. Though for our purposes, we only require regularity for global minimizers in the 2-torus, we establish the regularity result below, Proposition 2.1, for local minimizers in general n-dimensional domains since the regularity of (NLIP) is needed in other investigations such as [5] and we hope it will be useful to other authors. Given the well-developed regularity theory for area-minimizing sets, cf. e.g. [8], the issue here is to obtain a good estimate on the "excess-like" quantity that measures how far a set is from minimizing perimeter in a ball in terms of the radius of that ball, cf. (2.15). We show that even with the inclusion of the nonlocal term, it is still possible to obtain an estimate of order $\mathcal{O}(R^n)$ in a ball of radius R, hence allowing us to invoke the standard theory.

The proof of the main theorem appears in the third section. Our method for proving the global minimality of the lamellar critical point u_L consists of several steps. First, we must "corral" any reasonable competitor by showing that if the set where it equals one is not uniformly close to that of u_L, then necessarily, it has too much perimeter. Such a corralling step often occurs when working with sets of finite perimeter in the L^1-topology, cf. e.g. [15, 29], but here the argument is in some ways more subtle due to the nonlocal term in the energy which prefers multi-component competitors. Next, we show that competitors that are uniformly close are in fact C^2-close. Finally, we exploit the known stability of u_L in the sense of second variation to eliminate competitors that are C^2-close. For the scenario of converting stability to either local or global minimality in the context of (local) volume-constrained least area problems, we should mention the works [9, 16].

2 Regularity

In this section, we establish the regularity of the set $\partial\{x : u(x) = 1\}$ for any local minimizer u of the n-dimensional version of (NLIP), namely

$$\text{locally minimize} \quad \mathcal{E}_\gamma(A) := \int_U |\nabla \chi_A| + \gamma \int_U |\nabla v_A|^2 \, dx \tag{2.1}$$

over $A \subset U$ such that $|A| = m$, where $-\Delta v_A = u_A - m$ and $u_A := \chi_A - \chi_{A^c}$. Here U can either denote any bounded domain in $\mathbb{R}^n$ or the flat n-torus $\mathbb{T}^n$.

We note that the formulation above is of course equivalent to (NLIP) given by (1.1). We will phrase the regularity result in terms of L^1-local minimizers, by which we mean any set $\Omega \subset U$ of finite perimeter such that

$$\mathcal{E}_\gamma(\Omega) \leqslant \mathcal{E}_\gamma(A) \quad \text{provided} \int_U |\chi_\Omega - \chi_A| \, dx < \delta \tag{2.2}$$

for some $\delta > 0$.

Proposition 2.1. *Let Ω be an L^1-local minimizer of* (2.1) *and denote by $\partial^*\Omega$ the reduced boundary of Ω. Then $\partial^*\Omega \cap U$ is of class $C^{3,\alpha}$ for some $\alpha \in (0,1)$ and $\mathcal{H}^s[(\partial\Omega \setminus \partial^*\Omega) \cap U] = 0$ for every nonnegative s such that $s > n-8$. Furthermore, the solution v_Ω to $-\Delta v_\Omega = u_\Omega - m$ is of class $C^{1,\alpha}$ and one has the criticality condition*

$$(n-1)H(x) + 4\gamma v_\Omega(x) = \lambda \tag{2.3}$$

for any $x \in \partial^*\Omega \cap U$ where H denotes mean curvature of $\partial\Omega$, $\mathcal{H}^s$ denotes s-dimensional Hausdorff measure and λ is a constant.

Remark 2.2. *Condition* (2.3) *as well as the second variation of $\mathcal{E}_\gamma$ are derived in [5] under the assumption of regularity of the free boundary.*

Remark 2.3. *In Section 7.2 of [28], the authors establish $C^{3,\alpha}$ regularity of the reduced boundary of critical points of $\mathcal{E}_\gamma$ that arise in the limit $\varepsilon \to 0$ of critical points of the Ohta-Kawasaki energy $E_{\varepsilon,\gamma}$ given in* (1.3).

Proof. Let Ω be an L^1-local minimizer of $\mathcal{E}_\gamma$ and let x_0 be any point of $\partial\Omega \cap U$. Then let $D \subset\subset U$ be such that $x_0 \notin \overline{D}$ and

$$\int_D |\nabla \chi_\Omega| > 0.$$

By [7, Lemma 2.1], there exist two positive constants k_0 and l_0, depending only on D and $D \cap \Omega$, such that for every k, $|k| < k_0$, there exists a set F, with $F = \Omega$ outside D, and

$$\begin{gathered} |F| = |\Omega| + k, \\ \int_D |\nabla \chi_F| \leqslant \int_D |\nabla \chi_\Omega| + l_0|k|, \\ \int_D |\chi_F - \chi_\Omega| \, dx \leqslant l_0 |k| \int_D |\nabla \chi_\Omega|. \end{gathered} \tag{2.4}$$

Here when we write $|F|$, we mean the n-dimensional Lebesgue measure of the set F.

Fix $R > 0$ such that

$$\omega_n R^n < k_0, \quad \left(1 + l_0 \int_D |\nabla \chi_\Omega|\right) \omega_n R^n < \delta \quad \text{and} \quad \overline{B}_R(x_0) \cap \overline{D} = \emptyset, \tag{2.5}$$

where ω_n is the measure of the unit n-ball and δ comes from (2.2). Moreover, let $\tilde{F}$ minimize perimeter in $B_R(x_0)$ subject to the boundary values of Ω, i.e.

$$\int_{B_R(x_0)} |\nabla \chi_{\tilde{F}}| \leqslant \int_{B_R(x_0)} |\nabla \chi_F|$$

for all F such that $F \setminus B_R(x_0) = \Omega \setminus B_R(x_0)$. Without loss of generality, we can assume that $|\tilde{F} \cap B_R(x_0)| \leqslant |\Omega \cap B_R(x_0)|$. Since $\tilde{F} \cap \overline{D} = \Omega \cap \overline{D}$, we can use the same k_0 and l_0 as above with $\tilde{F}$ replacing Ω in (2.4). Hence, for $k := |\Omega| - |\tilde{F}| \leqslant \omega_n R^n < k_0$, there exists a set G, with $G = \tilde{F}$ outside D, and

$$|G| = |\Omega| = m, \tag{2.6}$$

$$\int_D |\nabla \chi_G| \leqslant \int_D |\nabla \chi_{\tilde{F}}| + CR^n, \tag{2.7}$$

$$\int_U |\chi_G - \chi_\Omega| \, dx \leqslant C_0 R^n < \delta, \tag{2.8}$$

where the last condition follows from (2.5) with $C_0 := \left(1 + l_0 \int_D |\nabla \chi_\Omega|\right) \omega_n$.

Since by (2.6), G is a competitor in (2.1), we have

$$\int_U |\nabla \chi_\Omega| + \gamma \int_U |\nabla v_\Omega|^2 \, dx \leqslant \int_U |\nabla \chi_G| + \gamma \int_U |\nabla v_G|^2 \, dx. \tag{2.9}$$

Thus, using the facts $\tilde{F} \setminus B_R(x_0) = \Omega \setminus B_R(x_0)$ and $G \setminus D = \tilde{F} \setminus D$, along with (2.7), the inequality (2.9) becomes

$$\begin{aligned}
\int_{U \setminus (D \cup B_R(x_0))} |\nabla \chi_\Omega| + \int_D |\nabla \chi_{\tilde{F}}| + \int_{B_R(x_0)} |\nabla \chi_\Omega| + \gamma \int_U |\nabla v_\Omega|^2 \, dx \\
\leqslant \int_{U \setminus (D \cup B_R(x_0))} |\nabla \chi_G| + \int_D |\nabla \chi_G| \\
+ \int_{B_R(x_0)} |\nabla \chi_G| + \gamma \int_U |\nabla v_G|^2 \, dx \\
= \int_{U \setminus (D \cup B_R(x_0))} |\nabla \chi_\Omega| + \int_D |\nabla \chi_G| \\
+ \int_{B_R(x_0)} |\nabla \chi_{\tilde{F}}| + \gamma \int_U |\nabla v_G|^2 \, dx \\
\leqslant \int_{U \setminus (D \cup B_R(x_0))} |\nabla \chi_\Omega| + \int_D |\nabla \chi_{\tilde{F}}| \\
+ \int_{B_R(x_0)} |\nabla \chi_{\tilde{F}}| + \gamma \int_U |\nabla v_G|^2 \, dx + CR^n.
\end{aligned}$$

Hence, we get

$$\int_{B_R(x_0)} |\nabla \chi_\Omega| - \int_{B_R(x_0)} |\nabla \chi_{\tilde{F}}| \leqslant \gamma \int_U |\nabla v_G|^2 \, dx - \gamma \int_U |\nabla v_\Omega|^2 \, dx + CR^n. \tag{2.10}$$

Now we estimate the nonlocal parts on the right-hand side of (2.10). To this end, let $w := v_\Omega - v_G$. Then $-\Delta w = u_\Omega - u_G$ with $\int_U w\,dx = 0$, where $|u_\Omega - u_G|$ is equal to zero in $U \setminus (B_R(x_0) \cup D)$ and is bounded by 2 in $B_R(x_0) \cup D$. Hence for any $p \geqslant 1$ we have

$$\|u_\Omega - u_G\|_{L^p(U)} \leqslant CR^{n/p}, \tag{2.11}$$

through an appeal to (2.8).

We take

$$p = \kappa \left(\frac{n}{n-1} \right), \tag{2.12}$$

where κ is less than but as close as needed to 1 so that $1 < p < \frac{n}{n-1}$ and apply the Calderon-Zygmund inequality (cf. e.g. [6, Chapter 9] or [11] for the periodic case and [31, Chapter 2] for the Neumann case),

$$\|w\|_{W^{2,p}(U)} \leqslant C\|u_\Omega - u_G\|_{L^p(U)}. \tag{2.13}$$

Then by the Sobolev imbedding theorem, for $q = \frac{\kappa n}{n-(1+\kappa)}$, along with Hölder's and the Poincaré inequality we get that

$$\|w\|_{L^1(U)} \leqslant C\|w\|_{L^q(U)} \leqslant C\|\nabla w\|_{L^q(U)} \leqslant C\|w\|_{W^{2,p}(U)}.$$

Note that, as $\kappa \to 1$, $\frac{\kappa n}{n-(1+\kappa)}$ approaches $\frac{n}{n-2}$, so in particular, $q > 1$. Thus, by combining (2.11)–(2.13) we obtain

$$\|w\|_{L^1(U)} \leqslant C\, R^{(n-1)/\kappa},$$

or in other words, since $\kappa < 1$, we have that

$$\int_U |v_\Omega - v_G|\,dx = \int_U |w|\,dx \leqslant C\, R^{n-1+\varepsilon} \tag{2.14}$$

for some $\varepsilon > 0$. (We should perhaps note that for the case $n = 2$, the desired inequality $\|w\|_{L^1(U)} \leqslant C\|u_\Omega - u_G\|_{L^p(U)}$ is a simple consequence of Poincaré and Hölder, without even needing an appeal to Calderon-Zygmund, since then H^1 imbeds continuously into any L^p with $p < \infty$.)

Now, using (2.8), (2.14) and integration by parts, we obtain the following bound on the difference of the nonlocal parts:

$$\begin{aligned}\int_U |\nabla v_G|^2\,dx - \int_U |\nabla v_\Omega|^2\,dx &\leqslant \int_U |u_\Omega - u_G|\,|v_G|\,dx + \int_U |v_\Omega - v_G|\,|u_\Omega|\,dx \\ &\leqslant C\, R^{n-1+\varepsilon}.\end{aligned}$$

Returning to (2.10), this implies that

$$\int_{B_R(x_0)} |\nabla \chi_\Omega| - \int_{B_R(x_0)} |\nabla \chi_{\tilde F}| \leqslant C\, R^{n-1+\varepsilon}. \tag{2.15}$$

Property (2.15) states that the boundary of the set Ω is almost area-minimizing in any ball. With this property in hand, the results of [14, 30] apply, and we can conclude that $\partial^*\Omega \cap U$ is of class $C^{1,\alpha}$, with $\mathcal{H}^s[(\partial\Omega \setminus \partial^*\Omega) \cap U] = 0$ for every $s > n - 8$.

Next we note that since $\|u_\Omega\|_{L^\infty} = 1$, we have a bound on $\|v_\Omega\|_{W^{2,p}}$ for any $p < \infty$ by the elliptic regularity. Hence, by the Sobolev imbedding theorem we get a bound on $\|v_\Omega\|_{C^{1,\alpha}}$ for any $\alpha < 1$ as well.

With $C^{1,\alpha}$ regularity of the reduced boundary in hand, one can locally describe $\partial^*\Omega$ in non-parametric form, as say, the graph of a $C^{1,\alpha}$ function Φ on a ball $B \subset \mathbb{R}^{n-1}$. Then one can compute the first variation of $\mathcal{E}_\gamma$ to find that Φ weakly solves

$$(n-1)\nabla \cdot \left(\frac{\nabla\Phi(x')}{\sqrt{1+|\nabla\Phi(x')|^2}} \right) = -4\gamma v_\Omega(x', \Phi(x')) + \lambda \quad \text{for } x' \in B$$

where λ appears as a Lagrange multiplier due to the volume constraint. Since the right-hand side is of class $C^{1,\alpha}$, it follows from standard elliptic theory that Φ is in fact of class $C^{3,\alpha}$ and so, in particular, (2.3) holds classically. □

3 Global Minimizers of E_γ

In this section, returning to the two-dimensional periodic setting, we will prove our main result, namely that for γ sufficiently small, the global minimizer u_γ of E_γ coincides with u_L, the lamellar critical point.

Here, emphasizing the γ dependence, we want to note that in two dimensions the criticality condition (2.3) obtained in Proposition 2.1 becomes

$$H_\gamma(x) + 4\gamma v_\gamma(x) = \lambda_\gamma \tag{3.1}$$

for all $x \in \partial\{x : u_\gamma(x) = 1\}$.

We can immediately conclude that any sequence of minimizers $\{u_\gamma\}$ of (1.1) converges, after perhaps a translation, to the lamellar minimizer u_L given by (1.4) of E_0 defined in (1.5).

Proposition 3.1. *For any m satisfying $|m| < 1 - \frac{2}{\pi}$, let $\{u_\gamma\}_{\gamma \geqslant 0}$ be a sequence of minimizers of E_γ. Then after perhaps a translation,*

$$u_\gamma \to u_L \quad \text{in } L^1(\mathbb{T}^2) \text{ as } \gamma \to 0. \tag{3.2}$$

Proof. Since a uniform bound $E_\gamma(u_\gamma) < C$ is immediate in light of the minimality of u_γ, one obtains a uniform BV-bound leading to convergence in L^1 of a subsequence. By the standard Γ-convergence argument, this limit must minimize E_0, the periodic isoperimetric problem. In two dimensions it is known that the global minimizer of E_0 is lamellar for $|m| < 1 - \frac{2}{\pi}$, hence is equal to u_L given by (1.4), after a translation, cf. [12, Section 7]. □

With this in hand, one can also easily establish convergence of the functions v_γ to v_L.

Proposition 3.2. *For any minimizer u_γ of E_γ, there is a value $\alpha \in (0,1)$ such that the corresponding solution v_γ of (1.2) is bounded in $C^{1,\alpha}$ independent of γ. Moreover, for a sequence of minimizers $\{u_\gamma\}_{\gamma \geqslant 0}$ of E_γ satisfying (3.2) we have*

$$v_\gamma \to v_L \quad \text{in} \quad H^2(\mathbb{T}^2).$$

In particular, $\int_{\mathbb{T}^2} |\nabla v_\gamma|^2 \, dx \to \int_{\mathbb{T}^2} |\nabla v_L|^2 \, dx$ *as* $\gamma \to 0$.

Proof. Note that since $\|u_\gamma\|_{L^\infty} = 1$, we have a γ-independent bound on $\|v_\gamma\|_{W^{2,p}}$ for any $p < \infty$ by the elliptic regularity. Hence, by the Sobolev imbedding theorem we get a γ-independent bound on $\|v_\gamma\|_{C^{1,\alpha}}$ for some $\alpha < 1$ as well. Similarly, $v_L \in W^{2,p}$.

Now, let $w_\gamma := v_\gamma - v_L$ and $\phi_\gamma := u_\gamma - u_L$. Note that $-\Delta w_\gamma = \phi_\gamma$ and $\phi_\gamma \to 0$ in L^p for all $1 \leqslant p < \infty$ as $\gamma \to 0$, since $\phi_\gamma \to 0$ in L^1 by assumption and $\|\phi_\gamma\|_{L^\infty} \leqslant 2$.

Since w_γ is periodic, through an integration by parts we obtain $\int_{\mathbb{T}^2} (\Delta w_\gamma)^2 \, dx = \int_{\mathbb{T}^2} |\nabla^2 w_\gamma|^2 \, dx$; hence, $\|\nabla^2 w_\gamma\|_{L^2} \to 0$ as $\gamma \to 0$.

Then, since $\int_{\mathbb{T}^2} v_\gamma \, dx = \int_{\mathbb{T}^2} v_L \, dx = 0$, we get that $\int_{\mathbb{T}^2} w_\gamma \, dx = 0$, and so the Poincaré inequality, $\|w_\gamma\|_{L^2} \leqslant C\|\nabla w_\gamma\|_{L^2}$ applies. Integrating by parts and using Hölder's inequality, we obtain

$$\|\nabla w_\gamma\|_{L^2}^2 = \int_{\mathbb{T}^2} w_\gamma \phi_\gamma \, dx \leqslant \|w_\gamma\|_{L^2} \|\phi_\gamma\|_{L^2} \leqslant C\|\nabla w_\gamma\|_{L^2} \|\phi_\gamma\|_{L^2},$$

and so $\|w_\gamma\|_{L^2}$ and $\|\nabla w_\gamma\|_{L^2}$ also tend to zero as $\gamma \to 0$. Thus, we get that $\|w_\gamma\|_{H^2} \to 0$. □

We now state our main result:

Theorem 3.3. *Fix any* m *such that* $|m| < 1 - \frac{2}{\pi}$. *Then for small* $\gamma > 0$, *the minimizers* $\{u_\gamma\}$ *of* E_γ *are lamellar, that is,* $u_\gamma \equiv u_L$ *up to translation.*

Proof. We will prove the theorem in several steps. For simplicity of presentation only, we will take $m = 0$ in the proof. Throughout the proof, we denote by S the strip $\{(x_1, x_2) : \frac{1}{4} < x_1 < \frac{3}{4},\ 0 < x_2 < 1\}$ and by Ω_γ the set $\{x : u_\gamma(x) = 1\}$.

Step 1. We first claim there cannot exist a sequence of components $S_\gamma^1 \subset \Omega_\gamma$ whose area converges to zero as $\gamma \to 0$.

To this end, we write Ω_γ as a union of its connected components, i.e. $\Omega_\gamma = \bigcup_{j=1}^{N_\gamma} S_\gamma^j$. We first note that necessarily, $N_\gamma < \infty$ since otherwise for fixed γ there would have to exist a sequence of components of Ω_γ whose area (and perimeter) approach zero. This would be impossible in light of (3.1) and Proposition 3.2 which imply a (γ-dependent) bound on the L^∞-norm of the curvature H_γ of $\partial\Omega_\gamma$.

Now we assume, by way of contradiction, that for a sequence of $\gamma-$values approaching zero, Ω_γ has a component, say S_γ^1, with $|S_\gamma^1| \to 0$.

Define $S_\gamma := \Omega_\gamma \setminus S_\gamma^1$. Then $\chi_{S_\gamma} \to \chi_S$ in L^1. Also, note that $\mathrm{Per}_{\mathbb{T}^2}(S_\gamma^1) \to 0$, for if not, that is, if say $c := \liminf_{\gamma\to 0} \mathrm{Per}_{\mathbb{T}^2}(S_\gamma^1)$ with $c > 0$, then since $\liminf \frac{1}{2}\mathrm{Per}_{\mathbb{T}^2}(S_\gamma) \geqslant \frac{1}{2}\mathrm{Per}_{\mathbb{T}^2}(S) = 1$, we get that

$$\liminf_{\gamma\to 0} E_\gamma(u_\gamma) \geqslant \frac{2+c}{2} > 1.$$

This yields a contradiction to the fact that $E_\gamma(u_\gamma) \to E_0(u_L) = \frac{1}{2}\mathrm{Per}_{\mathbb{T}^2}(S) = 1$ by Γ-convergence.

Now, the regularity of $\partial\Omega_\gamma$ asserted in Proposition 2.1 and the fact that $\mathrm{Per}_{\mathbb{T}^2}(S_\gamma^1) \to 0$ imply that we can enclose S_γ^1 in a disk whose radius approaches zero with γ. Shrink the disk until it touches ∂S^1_γ for the first time, denoting the radius of the shrunken disk by r_γ and the point where the disk touches ∂S^1_γ by p_γ. Then we have $H_\gamma(p_\gamma) \geqslant \frac{1}{r_\gamma}$ and so by evaluating the criticality condition (3.1) at $x = p_\gamma$, we see that $\lambda_\gamma \to \infty$ since $\|v_\gamma\|_{L^\infty}$ is bounded independent of γ by Proposition 3.2. Returning to (3.1) for $x \neq p_\gamma$, we conclude that in fact $H_\gamma(x) \to \infty$ for all $x \in \partial\Omega_\gamma$. Moreover, since $H_\gamma(p_\gamma) \geqslant \frac{1}{r_\gamma}$, for γ small enough, we have that, say, $H_\gamma(x) \geqslant \frac{1}{4r_\gamma}$ for all $x \in \partial\Omega_\gamma$ so S^j_γ is contained in a disk of radius $2r_\gamma$ for each $j \in \{1, \ldots, N_\gamma\}$.

For a finer analysis, let $\rho^j_\gamma := \operatorname{diam}(S^j_\gamma)$. Then S^j_γ is contained in a disk with radius ρ^j_γ. Now, define $\rho_\gamma := \min\{\rho^j_\gamma : j \in \{1, \ldots, N_\gamma\}\}$ so that

$$\operatorname{Per}_{\mathbb{T}^2}(S^j_\gamma) \geqslant \rho^j_\gamma \geqslant \rho_\gamma. \tag{3.3}$$

Then let the minimum ρ_γ be attained at, say, $j = j_0$, i.e., $\rho_\gamma = |p^{j_0}_\gamma - q^{j_0}_\gamma|$ with $p^{j_0}_\gamma$, $q^{j_0}_\gamma$ on $\partial S^{j_0}_\gamma$. As $\rho_\gamma = |p^{j_0}_\gamma - q^{j_0}_\gamma|$ and $S^{j_0}_\gamma$ is contained in a disk of radius ρ_γ which must be tangent to $\partial S^{j_0}_\gamma$, say, at $p^{j_0}_\gamma$, we see that $H_\gamma(p^{j_0}_\gamma) \geqslant \frac{1}{\rho_\gamma}$. Hence, using the L^∞-bound on v_γ and the criticality condition, we get that

$$\frac{1}{\rho_\gamma} - C_\gamma \leqslant H_\gamma(p^{j_0}_\gamma) + 4\gamma v_\gamma(p^{j_0}_\gamma) = \lambda_\gamma,$$

where C_γ depends only on γ and $\|v_\gamma\|_{L^\infty}$, and C_γ is $\mathcal{O}(\gamma)$. Thus at any point $x \in \partial\Omega_\gamma$ we have

$$H_\gamma(x) + C_\gamma \geqslant H_\gamma(x) + 4\gamma v_\gamma(x) = \lambda_\gamma \geqslant \frac{1}{\rho_\gamma} - C_\gamma,$$

and this gives that

$$H_\gamma(x) \geqslant \frac{1}{\rho_\gamma} - 2C_\gamma \geqslant \frac{1}{2\rho_\gamma}.$$

Thus for any $j \in \{1, \ldots, N_\gamma\}$, S^j_γ is contained in a disk of radius $2\rho_\gamma$. Using this fact we can find a lower bound on N_γ depending on ρ_γ as follows: Since $\frac{1}{2} = |\Omega_\gamma| = \sum_{j=1}^{N_\gamma} |S^j_\gamma| \leqslant \pi(2\rho_\gamma)^2 N_\gamma$, we get that

$$N_\gamma \geqslant \frac{1}{8\pi\rho^2_\gamma}.$$

This lower bound on N_γ with (3.3) will then imply that

$$\operatorname{Per}_{\mathbb{T}^2}(\Omega_\gamma) = \sum_{j=1}^{N_\gamma} \operatorname{Per}_{\mathbb{T}^2}(S^j_\gamma) \geqslant \rho_\gamma N_\gamma \geqslant \frac{1}{8\pi\rho_\gamma}.$$

Hence $\operatorname{Per}_{\mathbb{T}^2}(S_\gamma) \to \infty$ as $\gamma \to 0$, which contradicts the fact that $E_\gamma(u_\gamma) \to 1$.

Here we want to remark that the above argument also shows that there cannot be a sequence of components of the complement of Ω_γ approaching zero in measure.

Step 2. Let us again express Ω_γ as a union of its connected components, namely, $\Omega_\gamma = \bigcup_{j=1}^{N_\gamma} S^j_\gamma$. We next claim that for some j, the component S^j_γ has at least two noncontractible boundary components.

Note that if a component S^j_γ has one noncontractible boundary component, then since the homology classes of boundaries of a connected, compact set sum up to zero, there would then have to exist at least one other noncontractible boundary component of S^j_γ. In light of this, to establish our claim it suffices to reach a contradiction by assuming that all boundary components of Ω_γ are contractible in $\mathbb{T}^2$.

Considering the lifts of the boundary components to the covering space, we see that they are disjoint closed curves in $\mathbb{R}^2$, say, $\alpha_1, \ldots, \alpha_n$. After removing from this collection any curve α_i which is contained in the interior of the set enclosed by any other curve α_j, we can relist the remaining curves as $\alpha_1, \ldots, \alpha_k$, where $k \leqslant n$. It follows that all points not in the union of the interiors of $\alpha_1, \ldots, \alpha_k$ must either lie entirely in the lift of Ω_γ or else in the lift of Ω^c_γ. If, for example, the exterior of the curves lies completely within Ω^c_γ, then we have $|\Omega_\gamma| \leqslant \sum_{i=1}^k |\operatorname{int}\alpha_i|$; hence using the isoperimetric inequality in $\mathbb{R}^2$,

$$\left(\operatorname{Per}_{\mathbb{T}^2}(\Omega_\gamma)\right)^2 \geqslant \sum_{i=1}^k l^2(\alpha_i) \geqslant 4\pi \sum_{i=1}^k |\operatorname{int}\alpha_i| \geqslant 4\pi|\Omega_\gamma|,$$

where $\operatorname{int}\alpha_i$ denotes the interior of the set enclosed by α_i and $l(\alpha_i)$ denotes the length of the curve α_i. Similarly, if the the exterior of the curves $\alpha_1, \ldots, \alpha_k$ lies entirely in Ω_γ, then $|\Omega^c_\gamma| \leqslant \sum_{i=1}^k |\operatorname{int}\alpha_i|$, and so in that case we would have

$$\left(\operatorname{Per}_{\mathbb{T}^2}(\Omega_\gamma)\right)^2 = \left(\operatorname{Per}_{\mathbb{T}^2}(\Omega^c_\gamma)\right)^2 \geqslant \sum_{i=1}^k l^2(\alpha_i) \geqslant 4\pi \sum_{i=1}^k |\operatorname{int}\alpha_i| \geqslant 4\pi|\Omega^c_\gamma|.$$

Thus we get that

$$\left(\operatorname{Per}_{\mathbb{T}^2}(\Omega_\gamma)\right)^2 \geqslant 4\pi \min\{|\Omega_\gamma|, |\Omega^c_\gamma|\}.$$

Hence,

$$\liminf_{\gamma\to 0} \operatorname{Per}_{\mathbb{T}^2}(\Omega_\gamma) \geqslant \sqrt{2\pi} > 2,$$

which gives a contradiction since $E_\gamma(u_\gamma) \to E_0(u_L) = 1$. Here we have used $m = 0$ to see that $\min\{|\Omega_\gamma|, |\Omega^c_\gamma|\} = 1/2$. Note, however, that for any m such that $|m| < 1 - \frac{2}{\pi}$, one has $\min\{|\Omega_\gamma|, |\Omega^c_\gamma|\} > 1/\pi$, still yielding a contradiction. We point out that this is the only place in the proof where the restriction $|m| < 1 - \frac{2}{\pi}$ is used.

Therefore we see that there exists $j \in \{1, \ldots, N_\gamma\}$ such that S^j_γ has a boundary component which is not contractible in $\mathbb{T}^2$. Hence, it has at least two such boundary components. Let us denote these two noncontractible components of ∂S^j_γ by $\Sigma^j_{1,\gamma}$ and $\Sigma^j_{2,\gamma}$.

Step 3. We claim that $\Sigma^j_{1,\gamma}$ and $\Sigma^j_{2,\gamma}$ are both connected arcs on the unit square with endpoints either of the form $(a, 0)$, $(a, 1)$ or of the form $(0, a)$, $(1, a)$ for some $a \in [0, 1]$. Furthermore, we claim that Ω_γ consists of precisely one component with $\partial\Omega_\gamma = \Sigma^j_{1,\gamma} \cup \Sigma^j_{2,\gamma}$.

To this end, suppose $\Sigma^j_{1,\gamma}$ meets the boundary of the unit square $[0, 1] \times [0, 1]$ at, say, $(a, 0)$ or $(0, a)$ for some $a \in [0, 1]$. Then for it to be closed, its lift to the covering space must eventually pass through a point $(b, c) \in \mathbb{R}^2$ that differs from $(a, 0)$ or $(0, a)$ by an element of the lattice $\mathbb{Z}^2$, as $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. The shortest possibility is of course either $(a, \pm 1)$ or $(\pm 1, a)$, corresponding to vertical or horizontal line segments. However, failing this, the next shortest possibility would be either $(a \pm 1, \pm 1)$ or $(\pm 1, a \pm 1)$ and on the torus such a geodesic has length $\sqrt{2}$. This then would imply that $\mathrm{Per}_{\mathbb{T}^2}(\Sigma^j_{1,\gamma}) \geqslant \sqrt{2}$; hence, $\mathrm{Per}_{\mathbb{T}^2}(\Sigma^j_{1,\gamma} \cup \Sigma^j_{2,\gamma}) \geqslant 1 + \sqrt{2}$, contradicting the fact that $\mathrm{Per}_{\mathbb{T}^2}(\Omega_\gamma)$ must approach 2 as $\gamma \to 0$. This shows the first claim of Step 3.

Next we note that for all γ sufficiently small, Ω_γ cannot have other boundary components. If, for a sequence of γ-values approaching zero, $\partial\Omega_\gamma$ possessed another component with the corresponding sequence of perimeters having a positive limit, this would again result in $\liminf_{\gamma \to 0} \mathrm{Per}_{\mathbb{T}^2}(\Omega_\gamma) > 2$, an impossibility. On the other hand, Step 1 precludes the possibility of such a sequence of perimeters having zero limit.

Therefore Ω_γ has only one component with exactly two boundary components $\Sigma_{1,\gamma}$ and $\Sigma_{2,\gamma}$ having endpoints either of the form $(a, 0)$ and $(a, 1)$ or of the form $(0, a)$ and $(1, a)$ for some $a \in [0, 1]$.

Step 4. Now we will show that for some $C > 0$ independent of γ, one has

$$|H_\gamma(x)| \leqslant C\gamma \tag{3.4}$$

for all $x \in \partial\Omega_\gamma$.

We first claim that the constant λ_γ in (3.1) tends to zero as $\gamma \to 0$. To prove this claim, consider $\Sigma_{1,\gamma}$ as described above. Let $\theta_\gamma(s)$ denote the angle made between the tangent to $\Sigma_{1,\gamma}$ and the standard basis vector $(1, 0)$, where s denotes arc-length, and let l_γ denote the length of $\Sigma_{1,\gamma}$. From the regularity of $\Sigma_{1,\gamma}$ and its simple description from Step 3, and since $s = 0$ and $s = l_\gamma$ correspond to the same points on $\mathbb{T}^2$, the angles at the two endpoints, $\theta(0)$ and $\theta(l_\gamma)$, must agree. Then, since $H_\gamma = \frac{d\theta_\gamma}{ds}$, we can integrate (3.1) to see that

$$4\gamma \int_0^{l_\gamma} v_\gamma \, ds = \lambda_\gamma l_\gamma.$$

Noting that $l_\gamma \geqslant 1$ and invoking the uniform L^∞-bound on v_γ from Proposition 3.2, we get that $\lambda_\gamma \to 0$ as $\gamma \to 0$, and indeed, $\lambda_\gamma = \mathcal{O}(\gamma)$. Returning to (3.1), this immediately yields (3.4).

Step 5. We next claim that each boundary component of Ω_γ is globally the graph of a function.

It is enough to show this for $\Sigma_{1,\gamma}$, the arguments for the other component being identical. Since we are taking the lamellar set $S = \{x \in \mathbb{T}^2 : u_L(x) = 1\}$ to have vertical boundary components, we see that by Step 3, $\Sigma_{1,\gamma}$ passes through the points $(a, 0)$ and $(a, 1)$ because if it passed through $(0, a)$ and $(1, a)$ instead, while maintaining a curvature that is $\mathcal{O}(\gamma)$, this would yield a contradiction to the fact that $\chi_{\Omega_\gamma} \to \chi_S$ in L^1.

Consider the vertical line segments $\{0\} \times (0, 1)$ and $\{1\} \times (0, 1)$ and slide them to the right and left, respectively, until one of the translates touches $\Sigma_{1,\gamma}$ for the first time at a point $X^0 = (x_1^0, x_2^0)$ inside $(0, 1) \times (0, 1)$. Since $\Sigma_{1,\gamma}$ is smooth, in a neighborhood of X^0, $\Sigma_{1,\gamma}$ can be expressed locally as a graph of a function, say $x_2 \mapsto \frac{3}{4} + f_\gamma(x_2)$. In other words for some $\delta > 0$ the set $\{(\frac{3}{4} + f_\gamma(x_2), x_2) : x_2 \in (x_2^0 - \delta, x_2^0 + \delta)\}$ agrees with $\Sigma_{1,\gamma}$.

We claim that the graph of $\frac{3}{4} + f_\gamma(x_2)$ agrees with $\Sigma_{1,\gamma}$ for all $x_2 \in [0,1]$; in other words, we will show that the domain of f_γ can be extended to $[0,1]$. To establish this, let δ_0 be the smallest positive number such that $|f'_\gamma(x_2^0 + \delta_0)| \geqslant 1$. If no such number exists, then we can extend the domain of f_γ all the way to $x_2 = 1$. By the mean value theorem and the fact that $f'_\gamma(x_2^0) = 0$, we get

$$|f''_\gamma(\xi)| = \left| \frac{f'_\gamma(x_2^0 + \delta_0) - f'_\gamma(x_2^0)}{\delta_0} \right| \geqslant \frac{1}{\delta_0} > 1$$

for some $\xi \in (x_2^0, x_2^0 + \delta_0)$. This gives that

$$\left| \frac{f''_\gamma(\xi)}{(1 + (f'_\gamma(\xi))^2)^{3/2}} \right| > \frac{1}{2^{3/2}},$$

and we get a contradiction to the fact that $H_\gamma(f_\gamma(\xi), \xi)$ is arbitrarily close to 0 for γ small enough by Step 4. Therefore the domain of f_γ can be extended to $(x_2^0 - \delta, 1]$. Applying the same argument for the left-end point, we get that f_γ can be defined on the whole interval $[0,1]$. Similarly, the other boundary component $\Sigma_{2,\gamma}$ of Ω_γ coincides with the graph of $x_2 \mapsto \frac{1}{4} + g_\gamma(x_2)$ for some smooth function $g_\gamma : [0,1] \to \mathbb{R}$ and the claim follows.

Step 6. Now we will show that for small $\gamma > 0$, the minimizers $\{u_\gamma\}$ of E_γ are lamellar.

In light of Step 5, note that the set $\Omega_\gamma = \{x : u_\gamma(x) = 1\}$ takes the form

$$S_1 := \left\{ (x_1, x_2) \in \mathbb{T}^2 : \frac{1}{4} + g_\gamma(x_2) \leqslant x_1 \leqslant \frac{3}{4} + f_\gamma(x_2), \quad 0 \leqslant x_2 \leqslant 1 \right\}.$$

To begin, we note that without loss of generality, we may assume that

$$\int_0^1 f_\gamma \, dx_2 = \int_0^1 g_\gamma \, dx_2 = 0. \tag{3.5}$$

Indeed, translating the set S in the x_1-direction by the amount $\int_0^1 f_\gamma \, dx_2$ and then redefining the graphs f_γ and g_γ in terms of the deviation from this new vertical strip, we see that (3.5) follows, in light of the condition $\int_0^1 f_\gamma(x_2) - g_\gamma(x_2) \, dx_2 = 0$ which holds due to the mass constraint $|\Omega_\gamma| = 1/2$.

Now we will proceed to show that the global minimality of u_γ is violated for small γ unless $u_\gamma \equiv u_L$. To this end, for $t \in [0,1]$, define

$$S_t := \left\{ (x_1, x_2) \in \mathbb{T}^2 : \frac{1}{4} + t g_\gamma(x_2) \leqslant x_1 \leqslant \frac{3}{4} + t f_\gamma(x_2), \quad 0 \leqslant x_2 \leqslant 1 \right\}.$$

Note that, $\chi_{S_t} \to \chi_S$ in L^1 as $t \to 0$ and the mass constraint $\int_{\mathbb{T}^2} u_\gamma \, dx = 0$ along with (3.5) implies that $|S_t| = \frac{1}{2}$ for all $t \in [0,1]$. Hence, the family of functions

$$U(x,t) := \begin{cases} 1 & \text{if } x \in S_t, \\ -1 & \text{if } x \in S_t^c, \end{cases}$$

are all admissible competitors in the minimization of E_γ. We then let $V(x,t)$ be the solution of $-\Delta V(\cdot,t) = U(\cdot,t)$ subject to $\int_{\mathbb{T}^2} V(x,t)\,dx = 0$. Note that $U(x,0) = u_L(x)$, $U(x,1) = u_\gamma(x)$, $V(x,0) = v_L(x)$ and $V(x,1) = v_\gamma(x)$.

For $E_\gamma(U)(t) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla U(x,t)| + \gamma \int_{\mathbb{T}^2} |\nabla V(x,t)|^2\,dx$, define

$$e_\gamma(t) := E_\gamma(U)(t),$$

where $e_\gamma(0) = E_\gamma(u_L)$ and $e_\gamma(1) = E_\gamma(u_\gamma)$. Taylor's Theorem implies that

$$e_\gamma(1) = e_\gamma(0) + e'_\gamma(0) + \frac{1}{2}e''_\gamma(\tau) = e_\gamma(0) + \frac{1}{2}e''_\gamma(\tau) \tag{3.6}$$

for some $\tau \in (0,1)$ as u_L is a critical point of E_γ, making $e'_\gamma(0) = 0$.

Now we are going to calculate $e''_\gamma(\tau)$ explicitly.

Since

$$\frac{1}{2}\int_{\mathbb{T}^2} |\nabla U(x,t)| = \int_0^1 (1 + t^2(f'_\gamma(x_2))^2)^{1/2}\,dx_2 + \int_0^1 (1 + t^2(g'_\gamma(x_2))^2)^{1/2}\,dx_2,$$

a straight-forward calculation yields

$$\begin{aligned}\frac{d^2}{dt^2}\frac{1}{2}\int_{\mathbb{T}^2} |\nabla U(x,\tau)| = &\int_0^1 \Big\{(1+\tau^2(f'_\gamma(x_2))^2)^{-1/2}(f'_\gamma(x_2))^2 \\ &\qquad + (1+\tau^2(g'_\gamma(x_2))^2)^{-1/2}(g'_\gamma(x_2))^2\Big\}\,dx_2 \\ &- \tau^2\int_0^1 \Big\{(1+\tau^2(f'_\gamma(x_2))^2)^{-3/2}(f'_\gamma(x_2))^4 \\ &\qquad + (1+\tau^2(g'_\gamma(x_2))^2)^{-3/2}(g'_\gamma(x_2))^4\Big\}\,dx_2.\end{aligned}$$

Using the fact that $\tau \in (0,1)$ along with the conditions $(f'_\gamma(x_2))^2 < 1$ and $(g'_\gamma(x_2))^2 < 1$ by Step 5, one readily obtains the inequality

$$\frac{d^2}{dt^2}\frac{1}{2}\int_{\mathbb{T}^2} |\nabla U(x,\tau)| > \frac{1}{2\sqrt{2}}\left(\int_0^1 (f'_\gamma(x_2))^2\,dx_2 + \int_0^1 (g'_\gamma(x_2))^2\,dx_2\right). \tag{3.7}$$

Now using the definition of $U(x,t)$ and integrating by parts once, we can rewrite the nonlocal part of the energy as

$$\begin{aligned}\int_{\mathbb{T}^2} |\nabla V(x,t)|^2\,dx &= \int_{\mathbb{T}^2} U(x,t)V(x,t)\,dx \\ &= \iint_{S_t} V(x_1,x_2,t)\,dx_1dx_2 - \iint_{S_t^c} V(x_1,x_2,t)\,dx_1dx_2 \\ &= \int_0^1\int_{\frac{1}{4}+tg_\gamma(x_2)}^{\frac{3}{4}+tf_\gamma(x_2)} V(x_1,x_2,t)\,dx_1dx_2 \\ &\qquad - \int_0^1\int_{\frac{3}{4}+tf_\gamma(x_2)}^{\frac{1}{4}+tg_\gamma(x_2)} V(x_1,x_2,t)\,dx_1dx_2.\end{aligned}$$

Then taking two derivatives with respect to t, we find

$$\frac{d^2}{dt^2}\int_{\mathbb{T}^2}|\nabla V(x,\tau)|^2\,dx = 2\int_0^1\left\{V_{x_1}\left(\frac{3}{4}+\tau f_\gamma(x_2),x_2,\tau\right)f_\gamma^2(x_2)\right.$$
$$\left.-V_{x_1}\left(\frac{1}{4}+\tau g_\gamma(x_2),x_2,\tau\right)g_\gamma^2(x_2)\right\}dx_2$$
$$+2\int_0^1\left\{V_t\left(\frac{3}{4}+\tau f_\gamma(x_2),x_2,\tau\right)f_\gamma(x_2)\right.\tag{3.8}$$
$$\left.-V_t\left(\frac{1}{4}+\tau g_\gamma(x_2),x_2,\tau\right)g_\gamma(x_2)\right\}dx_2.$$

Let us concentrate on the second integral on the right-hand side of (3.8). Letting G denote the Green's function for the periodic Poisson problem, we get that

$$V\left(\frac{3}{4}+tf_\gamma(x_2),x_2,t\right) = \iint_{\mathbb{T}^2}G\left(\frac{3}{4}+tf_\gamma(x_2),x_2,y_1,y_2\right)U(y_1,y_2,t)dy_1dy_2$$
$$=\left(\iint_{S_t}-\iint_{S_t^c}\right)G\left(\frac{3}{4}+tf_\gamma(x_2),x_2,y_1,y_2\right)dy_1dy_2.$$

Similarly we have

$$V\left(\frac{1}{4}+tg_\gamma(x_2),x_2,t\right)=\left(\iint_{S_t}-\iint_{S_t^c}\right)G\left(\frac{1}{4}+tg_\gamma(x_2),x_2,y_1,y_2\right)dy_1dy_2.$$

Taking the derivative of $V\left(\frac{3}{4}+tf_\gamma(x_2),x_2,t\right)$ and $V\left(\frac{1}{4}+tg_\gamma(x_2),x_2,t\right)$ with respect to t and using the fact that $\int_{\mathbb{T}^2}G(x,y)dy=0$, we obtain

$$\int_0^1\left\{V_t\left(\frac{3}{4}+\tau f_\gamma(x_2),x_2,\tau\right)f_\gamma(x_2)\ -V_t\left(\frac{1}{4}+\tau g_\gamma(x_2),x_2,\tau\right)g_\gamma(x_2)\right\}dx_2=$$
$$=4\int_0^1\int_0^1\left\{G\left(\frac{3}{4}+\tau f_\gamma(x_2),x_2,\ \frac{3}{4}+\tau f_\gamma(y_2),y_2\right)f_\gamma(x_2)f_\gamma(y_2)\right.$$
$$-G\left(\frac{3}{4}+\tau f_\gamma(x_2),x_2,\frac{1}{4}+\tau g_\gamma(y_2),y_2\right)f_\gamma(x_2)g_\gamma(y_2)$$
$$-G\left(\frac{1}{4}+\tau g_\gamma(x_2),\ x_2,\frac{3}{4}+tf_\gamma(y_2),y_2\right)g_\gamma(x_2)f_\gamma(y_2)$$
$$\left.+G\left(\frac{1}{4}+\tau g_\gamma(\,x_2),x_2,\frac{1}{4}+\tau g_\gamma(y_2),y_2\right)g_\gamma(x_2)g_\gamma(y_2)\right\}dx_2dy_2.$$

Hence, the second integral on the right-hand side of the equation (3.8) becomes

$$4\int_{\partial S_\tau}\int_{\partial S_\tau}G(x,y)\zeta(x)\zeta(y)d\mathcal{H}^1(x)d\mathcal{H}^1(y),$$

where ζ is defined to be $f_\gamma(x_2)$ and $-g_\gamma(x_2)$ on the two components of ∂S_τ. However, by [13, Chapter 1], we have that

$$\int_{\partial S_\tau}\int_{\partial S_\tau} G(x,y)\zeta(x)\zeta(y)d\mathcal{H}^1(x)d\mathcal{H}^1(y) = \int_{\mathbb{T}^2}|\nabla\omega|^2 \geqslant 0, \tag{3.9}$$

where ω is an H^1 weak solution to the equation

$$-\Delta\omega = \mu \quad \text{on } \mathbb{T}^2,$$

and μ is the measure given by $\zeta\mathcal{H}^1\lfloor\partial S_\tau$.

Recall that $-\Delta V(x,t) = U(x,t)$, and since $|U| = 1$, it follows from elliptic regularity, that $\|V_{x_1}\|_{L^\infty} \leqslant C_0$ for some $C_0 > 0$, independent of t or γ. Hence, returning to (3.8), and dropping the second integral on the right-hand side, which has been shown to be positive by (3.9), we get that

$$\begin{aligned}\frac{d^2}{dt^2}\int_{\mathbb{T}^2}|\nabla V(x,\tau)|^2\,dx \geqslant 2\int_0^1 \Bigg\{ & V_{x_1}\left(\frac{1}{2}+\tau f_\gamma(x_2),x_2,\tau\right)f_\gamma^2(x_2) \\ & - V_{x_1}(\tau g_\gamma(x_2),x_2,\tau)g_\gamma^2(x_2)\Bigg\}\,dx_2 \\ \geqslant -2C_0 & \left(\int_0^1 f_\gamma^2(x_2)\,dx_2 + \int_0^1 g_\gamma^2(x_2)\,dx_2\right).\end{aligned} \tag{3.10}$$

Now combining the equations (3.7) and (3.10) with (3.6), and invoking (3.5) to apply the Poincaré inequality for f_γ and g_γ, we get that

$$\begin{aligned}e_\gamma(1) &= e_\gamma(0) + \frac{1}{2}e''_\gamma(\tau) \\ &\geqslant e_\gamma(0) + \frac{1}{4\sqrt{2}}\left(\int_0^1 (f'_\gamma(x_2))^2\,dx_2 + \int_0^1 (g'_\gamma(x_2))^2\,dx_2\right) \\ &\qquad - \gamma C_0\left(\int_0^1 f_\gamma^2(x_2)\,dx_2 + \int_0^1 g_\gamma^2(x_2)\,dx_2\right) \\ &\geqslant e_\gamma(0) + \left(\frac{\pi^2}{4\sqrt{2}} - \gamma C_0\right)\left(\int_0^1 f_\gamma^2(x_2)\,dx_2 + \int_0^1 g_\gamma^2(x_2)\,dx_2\right).\end{aligned}$$

For $\gamma < \frac{\pi^2}{4\sqrt{2}C_0}$, we conclude from the minimality of $E_\gamma(u_\gamma)$ $(= e_\gamma(1))$ that necessarily, $f_\gamma \equiv 0 \equiv g_\gamma$, that is, $u_\gamma \equiv u_L$. □

Acknowledgements. The authors were supported in this research by the National Science Foundation DMS-0654122. The authors would like to thank Rustum Choksi, Jiri Dadok, Chuck Livingston, Cyrill Muratov, Bruce Solomon and Yoshi Tonegawa for helpful conversations. We are grateful to Umberto Massari for pointing out to us Lemma 2.1 of [7] which plays a crucial role in the proof of regularity.

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