[9] R. Schoen and J. Wolfson, *Minimizing area among Lagrangian surfaces: the mapping problem*, J. Differential Geom. **58** (2001), 1–86.
[10] F. Schulze, *Uniqueness of compact tangent flows in mean curvature flow*, J. Reine Angew. Math. **690** (2014), 163–172.
[11] K. Smoczyk, *A canonical way to deform a lagrangian submanifold*, arXiv:dg-ga/9605005.
[12] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), 243–259.
[13] R. P. Thomas, *Moment maps, monodromy and mirror manifolds*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, (2001), 467–498.
[14] R. P. Thomas and S.-T. Yau, *Special Lagrangians, stable bundles and mean curvature flow*, Comm. Anal. Geom. **10** (2002), 1075–1113.

Nonlocal variational problems on polygons

Ihsan Topaloglu
(joint work with Marco Bonacini and Riccardo Cristoferi)

In [1] we consider the nonlocal isoperimetric problem

$$\inf\Big\{\mathcal{E}_\gamma(E)\,:\,|E|=1\Big\}, \tag{1}$$

among sets of finite perimeter $E\subset\mathbb{R}^d$ with given volume 1, where the energy functional $\mathcal{E}_\gamma$ is defined as

$$\mathcal{E}_\gamma(E) = \int_{\partial^* E}\psi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \gamma\int_E\int_E \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^\alpha}$$

for $\gamma>0$, $\alpha\in(0,d)$. We are interested in surface energies determined by *crystalline* surface tensions ψ, whose Wulff shapes (which are the corresponding isoperimetric regions) are given by convex polyhedra.

This minimization problem was recently introduced by Rustum Choksi, Robin Neumayer, and the author in [5] as an extension of the classical liquid drop model of Gamow [7, 4] to the anisotropic setting. In the *anisotropic* liquid drop model the competition is not only between the attractive and repulsive forces, but also between the anisotropy in the surface energy and the isotropy of the Riesz-like interaction energy. As in the isotropic case, the problem admits a minimizer when γ is sufficiently small and fails to have minimizers for large values of γ. However when ψ is smooth, and different than the Euclidean norm, its Wulff shape W_ψ is not a critical point of the energy $\mathcal{E}_\gamma(E)$ for any $\gamma>0$, whereas in the isotropic case the ball is the unique global minimizer for $\gamma>0$ sufficiently small.

In contrast, in [1], we prove that, for a wide class of crystalline surface tensions, where the Wulff shape of ψ enjoys particular symmetry properties, the corresponding isoperimetric set W_ψ *remains* as the minimizer of the nonlocal isoperimetric problem for small values of $\gamma>0$. To this end, let $\mathscr{P}_n$, $n\geq 3$, be the class of open, convex polygons $\mathcal{P}\subset\mathbb{R}^2$ with n sides $L_1,\dots,L_n$ and unit area $|\mathcal{P}|=1$, which are reflection symmetric with respect to the bisectors of all angles. Then our first main result states the minimality of polygons in $\mathscr{P}_n$.

Theorem 1. *Let $\mathcal{P} \in \mathscr{P}_n$ and let ψ be a surface energy density whose Wulff shape is $\mathcal{P}$. Then there exists $\bar{\gamma} > 0$, depending on $\mathcal{P}$ and α, such that for all $\gamma < \bar{\gamma}$ the polygon $\mathcal{P}$ is the unique (up to translations) solution to* (1)*.*

The proof of Theorem 1 follows by the combination of three main ingredients: (a) the stability of the Wulff inequality; (b) the fact that any solution to (1) is an ω-minimizer of the anisotropic perimeter and in turn, if γ is sufficiently small, it is a polygon with sides parallel to those of $\mathcal{P}$; and (c) the following quadratic upper bound for variations within the class $\mathscr{C}(\mathcal{P}, \varepsilon)$ where the sides of competitors are parallel to those of $\mathcal{P}$ and at distance at most ε.

Theorem 2 (Quadratic bound)**.** *Let $\mathcal{P} \in \mathscr{P}_n$. There exists $\varepsilon_0 > 0$ and $c_0 > 0$ (depending on the polygon $\mathcal{P}$ and on α) such that for every $\widetilde{\mathcal{P}} \in \mathscr{C}(\mathcal{P}, \varepsilon_0)$ one has the quadratic estimate*

$$\left| \int_{\mathcal{P}} \int_{\mathcal{P}} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^\alpha} - \int_{\widetilde{\mathcal{P}}} \int_{\widetilde{\mathcal{P}}} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^\alpha} \right| \leq c_0 |\mathcal{P} \triangle \widetilde{\mathcal{P}}|^2 \,.$$

For small γ, minimizers of $\mathcal{E}_\gamma$ are always obtained by perturbations of the Wulff shape of the surface energy, whose sides are translated parallel to themselves. In our result we exhibit an explicit class of Wulff shapes which remain global minimizers for $\gamma > 0$. However, we cannot prove that polygons in this class are exactly those with this global minimality property. It is an open problem to classify the crystalline anisotropies whose Wulff shapes remain the global minimizers of $\mathcal{E}_\gamma$ for $\gamma > 0$ sufficiently small which would require us to characterize the critical points of the nonlocal energy with respect to the restricted class of variations.

This question naturally led us to study a class of nonlocal repulsive energies of generalized Riesz-type

$$\mathcal{V}(E) \;=\; \int_E \int_E K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y$$

on polygons, where the kernel $K \geq 0$ is strictly decreasing and locally integrable.

The energy $\mathcal{V}$ (in any dimension) is uniquely maximized by the ball under volume constraint, as a consequence of Riesz's rearrangement inequality. Moreover, at least in the case of Riesz kernels, balls are characterized as the unique critical points for the energy under volume constraint. This was proved in a series of contributions (see e.g. [9]) via moving plane methods, and in full generality for Riesz kernels in [8] via a continuous Steiner symmetrization argument. In [2], we investigate the same two questions in a discrete setting, namely where the energy is evaluated on polygons with a fixed number of sides. In our first result we show that among triangles and quadrilaterals the regular polygon is the unique maximizer of the Riesz-type energy $\mathcal{V}$.

Theorem 3. *The equilateral triangle is the unique (up to rigid movements) maximizer of $\mathcal{V}$ in $\mathscr{P}_3$ under area constraint, and the square is the unique (up to rigid movements) maximizer of $\mathcal{V}$ in $\mathscr{P}_4$ under area constraint.*

The second main question that we address is whether the regular N-gon is characterized by the stationarity conditions, as it is the case for the ball. In order to state precisely this result, we need to fix some notation. Given two points $P, Q \in \mathbb{R}^2$, we denote by $\overline{PQ} = \{tP + (1-t)Q : t \in [0,1]\}$ the segment joining P and Q. For $N \geq 3$, let $\mathcal{P} \in \mathscr{P}_N$ be a polygon with N vertices $P_1, \ldots, P_N$. For notational convenience we also identify $P_0 = P_N$, $P_{N+1} = P_1$. For $i \in \{1, \ldots, N\}$ we let ℓ_i be the length of the side $\overline{P_i P_{i+1}}$, and M_i be the midpoint of the side $\overline{P_i P_{i+1}}$. Denoting by $v_{\mathcal{P}}(x) = \int_{\mathcal{P}} K(|x-y|)\,\mathrm{d}y$ the potential associated with the polygon, we then consider the following two conditions:

$$
\begin{aligned}
(2)\quad &\frac{1}{\ell_i}\int_{\overline{P_i P_{i+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) \\
&\qquad = \frac{1}{\ell_j}\int_{\overline{P_j P_{j+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) \quad \text{for all } i, j \in \{1, \ldots, N\},
\end{aligned}
$$

which corresponds to the criticality condition for the energy $\mathcal{V}$ under an area constraint, when sides are translated parallel to themselves, and

$$
\begin{aligned}
(3)\quad &\int_{\overline{P_i M_i}} v_{\mathcal{P}}(x)|x - M_i|\,\mathrm{d}\mathcal{H}^1(x) \\
&\qquad = \int_{\overline{P_{i+1} M_i}} v_{\mathcal{P}}(x)|x - M_i|\,\mathrm{d}\mathcal{H}^1(x) \quad \text{for all } i \in \{1, \ldots, N\},
\end{aligned}
$$

which corresponds to the criticality condition for the energy $\mathcal{V}$ under an area constraint, when a side is rotated around its midpoint. Our second result is the following.

Theorem 4. *If $\mathcal{P} \in \mathscr{P}_3$ obeys condition* (3)*, then $\mathcal{P}$ is an equilateral triangle. If $\mathcal{P} \in \mathscr{P}_4$ obeys conditions* (2) *and* (3)*, then $\mathcal{P}$ is a square.*

The proof of this theorem uses a reflection argument inspired by [6] as well as an argument based on a continuous symmetrization, inspired by an idea of Carrillo, Hittmeir, Volzone, and Yao [3]. We prove that the conditions (2) and (3) enforce the property of being equilateral, thus reducing the proof to the class of rhombi; then in a second step we prove that the polygon has to be also equiangular, using a reflection argument.

References

[1] M. Bonacini, R. Cristoferi, and I. Topaloglu, *Minimality of polytopes in a nonlocal anisotropic isoperimetric problem*, Nonlinear Anal. **205** (2021), Paper No. 112223.

[2] M. Bonacini, R. Cristoferi, and I. Topaloglu, *Riesz-type inequalities and overdetermined problems for triangles and quadrilaterals*, J. Geom. Anal. **32** (2022), Paper No. 48.

[3] J. A. Carrillo, S. Hittmeir, B. Volzone, and Y. Yao, *Nonlinear aggregation-diffusion equations: radial symmetry and long time asymptotics*, Invent. Math. **218** (2019), 889–977.

[4] R. Choksi, C. B. Muratov, and I. Topaloglu, *An old problem resurfaces nonlocally: Gamow's liquid drops inspire today's research and applications*, Notices Amer. Math. Soc. **64** (2017), 1275–1283.

[5] R. Choksi, R. Neumayer, and I. Topaloglu, *Anisotropic liquid drop models*, Adv. Calc. Var. **15** (2022), 109–131.

[6] I. Fragalà and B. Velichkov, *Serrin-type theorems for triangles*, Proc. Amer. Math. Soc. **147** (2019), 1615–1626.

[7] G. Gamow, *Mass defect curve and nuclear constitution*, Proc. R. Soc. Lond. A **126** (1930), 632–644.
[8] J. Gómez-Serrano, J. Park, J. Shi, and Y. Yao, *Symmetry in stationary and uniformly rotating solutions of active scalar equations*, Duke Math. J. **170** (2021), 2957–3038.
[9] G. Lu and J. Zhu, *An overdetermined problem in Riesz-potential and fractional Laplacian*, Nonlinear Anal. **75** (2012), 3036–3048.

An L^1 method for convergence and metastability

Maria G. Westdickenberg
(joint work with Sarah Biesenbach, Felix Otto, Sebastian Scholtes, and Richard Schubert)

We present a novel method developed for the 1-d Cahn Hilliard equation in [6] and extended in [1]. In the first part of the talk, we consider the one-dimensional, fourth-order Cahn–Hilliard equation

$$u_t = -\big(u_{xx} - G'(u)\big)_{xx} \qquad t > 0, \qquad x \in \mathbb{R}, \tag{1}$$

where G is a double-well potential with nondegenerate absolute minima at ± 1; a canonical choice is $G(u) = \frac{1}{4}(1-u^2)^2$.

The Cahn–Hilliard equation has a gradient flow structure with energy and dissipation given by

$$E(u) = \int \frac{1}{2}u_x^2 + G(u)\,dx, \qquad D := \int \Big(\big(G'(u) - u_{xx}\big)_x\Big)^2\,dx. \tag{2}$$

The so-called "centered kink" v minimizes the energy subject to ± 1 boundary conditions at $\pm\infty$ and is normalized so that $v(0) = 0$. We call its energy $e_* := E(v)$. For any $a \in \mathbb{R}$, the kink $v_a := v(\cdot - a)$ is also an energy minimizer, so that there is a whole continuum of minima.

We are interested in optimal convergence rates for initial data that is an order-one L^1 perturbation of a kink. It turns out that it is useful to work in terms of the L^2-closest kink $v_c(x) = v(x-c)$, the shift c, and the associated L^1 distance

$$V := \int |u - v_c|\,dx.$$

Further defining the energy-gap $\mathcal{E} := E(u) - E(v)$, we are able to establish a Nash-type estimate

$$\mathcal{E} \lesssim D^{\frac{1}{3}}(V+1)^{\frac{4}{3}},$$

from which an elementary ODE argument yields

$$\mathcal{E} \lesssim \frac{\bar{V}^2 + 1}{t^{\frac{1}{2}}} \qquad \text{for } t \in [0,T], \qquad \text{where } \bar{V} := \sup_{t \le T} V. \tag{3}$$

Hence it remains "only" to deduce that V remains bounded. For this, we use a duality argument inspired by [5] together with decay estimates for the linear equation on a domain with a (subcritical) moving boundary and a buckling argument.