ANISOTROPIC LIQUID DROP MODELS

RUSTUM CHOKSI, ROBIN NEUMAYER, AND IHSAN TOPALOGLU

Abstract. We introduce and study certain variants of Gamow's liquid drop model in which an anisotropic surface energy replaces the perimeter. After existence and nonexistence results are established, the shape of minimizers is analyzed. Under suitable regularity and ellipticity assumptions on the surface tension, Wulff shapes are minimizers in this problem if and only if the surface energy is isotropic. In sharp contrast, Wulff shapes are the unique minimizers for certain crystalline surface tensions. We also introduce and study several related liquid drop models with anisotropic repulsion for which the Wulff shape is the minimizer in the small mass regime.

1. Introduction

Gamow's liquid drop (LD) model, early versions of which date back to 1930 ([29]), has recently generated considerable interest in the calculus of variations community (see [16] for a general introduction). It was initially developed to predict the mass defect curve and the shape of atomic nuclei. In its modern rendition, it includes two competing forces: an attractive surface energy associated with a depletion of nucleon density near the nucleus boundary, and repulsive Coulombic interactions due to the presence of positively charged protons. Mathematically, it has a very simple form: over all sets $E \subset \mathbb{R}^3$ of measure m, minimize

$$\mathcal{P}(E) + \int_E \int_E \frac{dxdy}{|x-y|},$$

where $\mathcal{P}(E)$ denotes the perimeter in the geometric measure-theoretic sense. As such, the LD model is a paradigm for shape optimization via competitions of short and long-range interactions, and indeed it (or variants of it) has been used to model many different systems at all length scales, from atomic (its original conception) to cosmological.

Often studied is the generalization of the LD model in which one works in n space dimensions with any Riesz potential; that is, for fixed $\alpha \in (0, n)$ we consider the variational problem

$$\inf \Big\{ \mathcal{E}(E) := \mathcal{P}(E) + \mathcal{V}(E) \,\Big|\, |E| = m \Big\} \quad \text{where} \quad \mathcal{V}(E) := \int_E \int_E \frac{dxdy}{|x-y|^\alpha}. \tag{1.1}$$

From a mathematical point of view, the LD model has two notable features:

Feature (i). Balls are extremal for each individual term but at opposite ends of the spectrum – balls are best for (minimizers of) the first term but worst for (maximizers of) the second term. In particular, a ball of mass m is always a critical point of $\mathcal{E}$ among volume-preserving variations.

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Feature (ii). The two terms scale differently in mass m, with perimeter dominating for small mass and repulsion dominating for large mass.

In [15], it is conjectured that up to a critical mass m_c, balls are the unique minimizers, while above m_c, minimizers fail to exists. Due primarily to the work of Knüpfer and Muratov ([36, 37]) and Figalli et al. ([21]), with additional/related contributions from [9, 14, 28, 35, 40], the state of the art for (global) minimizers of (1.1) is as follows: For any $n \geqslant 2$, we have:

(G1) for all $\alpha \in (0, n)$ there exists $\widetilde{m}_1 > 0$ such that if $m \leqslant \widetilde{m}_1$, then the problem admits a minimizer;

(G2) for all $\alpha \in (0, n)$ there exists $\widetilde{m}_0 > 0$, $\widetilde{m}_0 \leqslant \widetilde{m}_1$, such that if $m \leqslant \widetilde{m}_0$, then the minimizer is uniquely (modulo translations) given by the ball of mass m; and

(G3) for all $\alpha \in (0, 2)$ there exists $\widetilde{m}_2 > 0$ such that if $m > \widetilde{m}_2$, then no minimizer exists.

It is conjectured in [15] that $\widetilde{m}_0 = \widetilde{m}_1 = \widetilde{m}_2$ when $n = 3$ and $\alpha = 1$. While the conjecture remains open, it was shown in [9] that $\widetilde{m}_0 = \widetilde{m}_1 = \widetilde{m}_2$ in any dimension for α sufficiently small. It also remains open whether the nonexistence result (G3) can be extended to $\alpha \in [2, n)$.

In this article we introduce and discuss anisotropic variants of (1.1). In particular, we address two classes of anisotropic liquid drop models consisting of

(1) anisotropic perimeter with isotropic long-range repulsions;

(2) anisotropic perimeter with *related* anisotropic long-range repulsions.

To our knowledge this is the first mathematical treatment of these problems, particularly surprising since they are both physically and mathematically well-motivated. On the physical side, it is natural to consider surface energies which are anisotropic (cf. [41]). Indeed, at the microscopic level, the existence of a tensor force can produce an asymmetry in the nucleon-nucleon potential, creating an anisotropic surface tension. From the more macroscopic perspective, surface diffuseness can vary across the interface boundary, also creating an anisotropic surface tension. In such situations, it is natural to couple the anisotropic surface energy with an isotropic (e.g. Coulombic) long-range interactions due to the presence of charged particles. Thus we arrive at class (1). On the other hand, atomic lattice structures as seen, for example, in Ising spin systems, can have competing anisotropic magnetic interactions (see for example [30, 32]). For example, ferromagnetic Ising models can have anisotropic interactions that are weighted towards one of the principle lattice axes.

Mathematically, it is natural to consider anisotropic LD models because of the richer interaction between the features (i) and (ii). The mass scaling feature (ii) prevails and, hence, in Theorem 3.1 of Section 3 we readily establish existence for small mass and nonexistence for large mass in direct analogy to parts (G1) and (G3) above. Our proof combines several techniques used in the literature in a novel way. However, feature (i), wherein the ball is naturally replaced by the Wulff shape associated with the anisotropy, is *subtle*: While the Wulff shape is minimal for the perimeter term, its relation to the second term is, in certain cases, unclear. Thus what is fundamentally different for these anisotropic LD models is the structure of minimizers for small mass regime. As we show in this article, this question is rich and, indeed, our work opens up far more questions than it solves. We now present and discuss our results for each class of models, and in doing so, explicitly state the main theorems of this article.

1.1. **Class I: Anisotropy in the Surface Energy.** Consider a *surface tension*

$$f : \mathbb{R}^n \to [0, \infty)$$

to be positively one-homogeneous, convex, and positive on $\mathbb{R}^n \setminus \{0\}$. For a set of finite perimeter E, we let

$$\mathcal{P}_f(E) := \int_{\partial^* E} f(\nu_E)\, d\mathcal{H}^{n-1}$$

be the associated anisotropic surface energy, where $\partial^* E$ denotes the reduced boundary, ν_E the measure-theoretic outer unit normal, and $\mathcal{H}^{n-1}$ the $(n-1)$ dimensional Hausdorff measure. Our first class of anisotropic LD models is given by

$$\inf \Big\{ \mathcal{E}_f(E) := \mathcal{P}_f(E) + \mathcal{V}(E) \;\Big|\; |E| = m \Big\}, \tag{1.2}$$

where $\mathcal{V}$ is defined as in (1.1). When $f(\cdot) = |\cdot|$ (the Euclidean norm), our problem (1.2) reduces to the LD problem (1.1).

Let us recall that the global minimizer of the anisotropic isoperimetric problem

$$\inf \big\{ \mathcal{P}_f(E) \,\big|\, |E| = m \big\}$$

is (a dilation or translation of) the *Wulff shape* K of f where

$$K := \bigcap_{\nu \in \mathbb{S}^{n-1}} \big\{ x \in \mathbb{R}^n \,\big|\, x \cdot \nu < f(\nu) \big\}; \tag{1.3}$$

see [10, 24, 25, 48, 49]. The Wulff shape is a bounded convex set that plays the role of the *ball* in the anisotropic setting.

Given that the anisotropic perimeter dominates the nonlocal term for small mass, one might initially be tempted to infer that, in this regime, the perimeter completely determines the behavior of minimizers, and minimizers are the Wulff shape for small mass. On the other hand, this would require Wulff shapes to be critical points of the *isotropic* repulsion. It turns out that the Euler-Lagrange equation serves as an obstruction for the minimality of the Wulff shape when f satisfies certain smoothness and ellipticity conditions, but not in general: the Wulff shape can be a minimizer of (1.2). As we show, the role of the Wulff shape *depends crucially on the regularity and ellipticity of f.* To this end, let us introduce two important classes of surface tensions. We say that f is a *smooth elliptic surface tension* if $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and there exist constants $0 < \lambda \leqslant \Lambda < \infty$ such that for every $\nu \in \mathbb{S}^{n-1}$,

$$\lambda\, |\tau|^2 \leqslant \nabla^2 f(\nu)[\tau, \tau] \leqslant \Lambda\, |\tau|^2$$

for all $\tau \in \mathbb{R}^n$ with $\tau \cdot \nu = 0$. For such surface tensions, the corresponding Wulff shape has C^∞ boundary and is uniformly convex. We say that f is a *crystalline* surface tension if for some N finite and $x_i \in \mathbb{R}^n$,

$$f(\nu) = \max_{1 \leqslant i \leqslant N} x_i \cdot \nu.$$

For crystalline surface tensions, the corresponding Wulff shape K is a convex polyhedron.

One of the main contributions of this paper is the following two contrasting Theorems: one general result for smooth surface elliptic tensions f and a sharply contrasting example for a crystalline surface tension in 2D.

Theorem 1.1. *Let $n \geqslant 2$ and $m > 0$. Let f be a smooth elliptic surface tension with Wulff shape K. Then we have the following two statements.*

(i) *Suppose*

$$\alpha \in \left(0, n - \frac{1}{3}\right).$$

Then K is a critical point of (1.2) if and only if f is the Euclidean norm.

(ii) *Suppose*

$$\alpha \in \left(0, n - (\sqrt{2} - 1)\right).$$

Then there exists $\bar{m}_1$ depending on $n, f,$ and α such that the following holds. Suppose E is a minimizer of (1.2) for mass $m \leqslant \bar{m}_1$. Then for no other mass m a dilation of E is even a critical point, unless f is the Euclidean norm and E is a ball.

When surface tensions (e.g. crystalline) lack these smoothness and ellipticity properties, it no longer makes sense to write down the Euler-Lagrange equation of (1.2) for arbitrary (smooth, compactly supported) variations. This means that the analysis for Theorem 1.1 cannot be extended to this case. However, this is not purely a technical issue: Indeed, in contrast to the smooth elliptic case, we have the following result in the crystalline case.

Theorem 1.2. *Let $n = 2$, $\alpha \in (0, 2)$, and let f be the surface tension*

$$f(\nu) = \frac{1}{2}\|\nu\|_{\ell^1(\mathbb{R}^2)} = \frac{1}{2}(|\nu \cdot e_1| + |\nu \cdot e_2|), \tag{1.4}$$

whose corresponding Wulff shape K is the square $[-1/2, 1/2] \times [-1/2, 1/2]$ of volume one. There exists $\bar{m}_2$ depending on α such that for $m \leqslant \bar{m}_2$ the Wulff shape is the unique (modulo translations) minimizer of (1.2).

Let us provide a few comments on these theorems and their proofs. Theorem 1.1 sheds considerable light on the case (G2), which states that balls minimize (1.1) for small masses. It is tempting to interpret this result as a consequence of scaling: for sufficiently small mass, the perimeter term dominates the nonlocal term and completely governs the behavior of minimizers, and hence the minimizers are balls. Theorem 1.1(i) shows that this is not the case, since the energies in (1.1) and (1.2) scale in the same way. Moreover, in contrast to the classical liquid drop model where for every mass below a certain threshold, the minimizer of (1.1) is just a dilation of the same set (i.e., the ball), if E is a minimizer of (1.2) for suitably small mass, then a dilation of E cannot be critical for any other mass. A natural question then remains as to the nature of minimizers. To this end, we do present some partial results in Theorem 2.2 by showing that a minimizer is a small uniformly convex perturbation of the Wulff shape. A natural question then remains as to the nature of minimizers. After rescaling, the boundaries of minimizers converge in the Hausdorff topology to the boundary of the Wulff shape as $m \to 0$. For smooth and elliptic surface tensions, the regularity theory then implies that rescaled minimizers converge smoothly to the Wulff shape (in particular, they are uniformly convex for m sufficiently small).

See Section 2 for more details. It is not clear if one could expect to give an explicit characterization of minimizers.

The proof of Theorem 1.1 is based upon an analysis of the first variation of (1.2). Besides regularity of minimizers (established in Theorem 2.2) and a characterization of sets with constant first variation of $\mathcal{P}_f$ in [34], the main tool needed to prove Theorem 1.1 is the fact that only balls have constant first variation of $\mathcal{V}$. While this was known for $\alpha \in (0, n-1)$, we produce more delicate arguments to extend the result to $\alpha \in [n-1, n-\frac{1}{3})$. To this end, we use a moving planes argument to prove the following:

Theorem 1.3. *Fix $n \geqslant 2$ and $\alpha \in (0, n-1/3)$. Suppose that $E \subset \mathbb{R}^n$ is a bounded domain with ∂E of class*

$$\begin{aligned} &C^1 \quad \text{if } \alpha < n-1\,, \\ &C^{1,\gamma} \text{ with } 1+\gamma > 1/(n-\alpha) \text{ if } \alpha \in [n-1, n-1/2)\,, \\ &C^{2,\gamma} \text{ with } 2+\gamma > 1/(n-\alpha) \text{ if } \alpha \in [n-1/2, n-1/3)\,. \end{aligned}$$

Let $v_E(x)$ be the Riesz potential

$$v_E(x) = \int_E \frac{dy}{|x-y|^\alpha}.$$

If v_E is constant on ∂E, then E is a ball.

Our Riesz potential restrictions $\alpha \in (0, n-1/3)$ in Theorem 1.3 and Theorem 1.1 (i), as well as the requirement $\alpha \in \big(0, n-(\sqrt{2}-1)\big)$ for Theorem 1.1 (ii), warrant the following remark.

Remark 1.4 (*The range of Riesz potentials*). Theorem 1.3 was established for the Coulombic case $\alpha = n-2$ in [26] and was extended to $\alpha \in (0, n-1)$ in [39], both using the method of moving planes; see also [45]. The case when $\alpha \geqslant n-1$ is significantly more delicate, principally due to the fact that the Riesz potential v_E is merely Hölder continuous in this case; see (2.5).

Our proof of Theorem 1.3 in the subtler case $\alpha \in [n-1, n-1/3)$ pairs the method of moving planes on integral forms in the spirit of [13, 39] with some new reflection arguments and estimates on how the Riesz potential grows compared to its reflection across a hyperplane.

In order to apply Theorem 1.3 to Theorem 1.1 (i) and (ii), we need to establish, respectively, regularity for the Wulff shape K and for a minimizer E of (1.2) for mass m. The regularity of the Wulff shape K depends on that of the elliptic surface tension which we have conveniently assumed to be C^∞. With this smoothness assumption, we have sufficient regularity to directly employ Theorem 1.3. On the other hand, the regularity result for minimizers of E (Theorem 2.2) gives a further restriction on α, yielding the assumption $\alpha \in \big(0, n-(\sqrt{2}-1)\big)$.

After the submission of this article, Gómez-Serrano, Park, Shi, and Yao extended Theorem 1.3 in [33, Theorem C] to the full range $\alpha \in (0, n)$ using continuous Steiner symmetrization. Furthermore, their proof of Theorem 1.3 applies to sets with Lipschitz regular boundaries. With this result in hand, one can remove the technical assumptions from Theorem 1.1(i) and (ii) to extend the results to all $\alpha \in (0, n)$.

Our Theorem 1.2 is in contrast to the smooth elliptic setting, and together with Theorem 1.1 demonstrates an interesting situation where the regularity and ellipticity of the surface tension govern a fundamental aspect of the problem: whether the isoperimetric set is the minimizer of (1.2). Typically in anisotropic isoperimetric problems, the regularity and ellipticity of the surface tension affect quantitative aspects of the problem (for instance, regularity of quasi-minimizers), but not qualitative aspects of the problem. Theorem 1.2 should be regarded as an example (or counter-example), and not generic for crystalline surface tensions. It is crucially based upon a 2D result of Figalli and Maggi (cf. Theorem 5.1) which proves that quasi-minimizers of crystalline anisotropic surface energies must be convex polygons. Minimizers of (1.2) are quasi-minimizers of the anisotropic perimeter, so this effectively transforms (1.2) to a finite dimensional problem. For the simple case of a square Wulff shape, one can explicitly calculate $\mathcal{V}$. While we believe the result holds true in 2D for Wulff shapes that are regular polygons, our calculation uses the symmetries of the Wulff shape given by the dihedral group of order 8. Thanks to a recent result by Figalli and Zhang posted after the submission of this article (see [23, Theorem 1.1]), it is also possible to extend our Theorem 1.2 to higher dimensions where the Wulff shape is given by a cube (cf. Remark 5.2).

We remark that the subtleties of addressing (1.2) for crystalline surface tensions highlights the lack of a general theory in the modern calculus of variations to address extremal notions, like criticality, for nondifferentiable, nonconvex functionals. A key point becomes understanding among what variations one can compute the Euler-Lagrange equation and whether one can derive meaningful information from computing first and second variations among a restricted class of variations. This question is an important one in the setting of crystalline mean curvature flow; see, among others, [4, 5, 6, 7, 12, 11, 50, 51] and references therein. The paper [18] also investigates this theme.

1.2. **Class II: Anisotropy in the Surface Energy and the Repulsive Term.** In light of Theorem 1.1 and the physical motivation, it would seem natural to replace the repulsive term $\mathcal{V}(E)$ with f-driven anisotropic interactions which are maximized (under fixed volume) by the Wulff shape K. To this end, let us assume the surface tension f is smooth and elliptic and denote by f_* the dual to f defined by

$$f_*(x) = \sup\{x \cdot \nu \,\big|\, f(\nu) \leqslant 1\}\,.$$

Note that the Wulff shape K can be equivalently expressed as the unit ball for f_*, that is, $K = \{x : f_*(x) < 1\}$. We consider three classes of variational problems in the spirit of (1.2):

$$\inf\Big\{\mathcal{P}_f(E) + \mathcal{U}_1(E) \,\Big|\, |E| = m\Big\} \tag{1.5}$$

and

$$\inf\Big\{\mathcal{P}_f(E) + \mathcal{U}_i(E) \,\Big|\, |E| = m,\ E \subset B_{c_{n,f} m^{1/n}}\Big\}\,, \qquad i = 2, 3, \tag{1.6}$$

for some $c_{n,f}$ depending on n and f, where we let

$$\begin{aligned} \mathcal{U}_1(E) &:= \sup_{y\in\mathbb{R}^n} \int_E f_*(x-y)^{-\alpha}\,dx \qquad \text{for } \alpha\in(0,1) \\ \mathcal{U}_2(E) &:= -\inf_{y\in\mathbb{R}^n} \int_E f_*(x-y)^{\beta}\,dx \qquad \text{for } \beta\in(0,\infty)\,, \\ \mathcal{U}_3(E) &:= -\inf_{y\in\mathbb{R}^n} \int_E \log(f_*(x-y))\,dx\,. \end{aligned} \tag{1.7}$$

The confinement constraint in (1.6) is needed, otherwise the infimum is minus infinity for all m and so no minimizer exists. As we show in Section 6, each $\mathcal{U}_i(E)$ in (1.7) is maximized by the Wulff shape among sets of a fixed volume, so the variational problems (1.5) and (1.6) exhibit *both* of the analogous two features of the isotropic LD model (1.1). In Section 6 we prove the following theorem.

Theorem 1.5. *Let $n \geqslant 2$ and let f be a smooth elliptic surface tension. There exists a constant $m' = m'(n, f, \mathcal{U}_i)$ such if $m \leqslant m'$, any minimizer of* (1.5) *or* (1.6) *is a Wulff shape.*

The main tool in Theorem 1.5 is a strong form of the quantitative Wulff inequality from [43]. Our method of proof is flexible; the two key ingredients are the subcritical scaling of $\mathcal{U}_i$ with respect to the surface energy and the criticality of K, and one can adapt the proof to other functionals satisfying these properties.

Remark 1.6 (*Equilibrium Figures à la Poincaré with an Anisotropic Potential*). Perhaps the most natural way to incorporate anisotropic repulsions would be to replace $\mathcal{V}(E)$ with

$$\mathcal{V}_f(E) = \int_E \int_E \frac{1}{f_*(x-y)^{\alpha}}\,dxdy\,, \tag{1.8}$$

and consider the minimization problem

$$\inf\Big\{\mathcal{P}_f(E) + \mathcal{V}_f(E)\,\Big|\,|E| = m\Big\}. \tag{1.9}$$

Apart from the trivial case where $f(\nu) = |A\nu|$ for a positive definite matrix A (where a linear change of variables transforms (1.9) into the isotropic LD model (1.1)), we do not know whether the Wulff shape is the minimizer for (1.9) for small mass.

The issue is related to the anisotropic problem

$$\inf\Big\{-\mathcal{V}_f(E)\,\Big|\,|E| = m\Big\}. \tag{1.10}$$

In the isotropic case (f being the Euclidean norm) and $\alpha = n-2$ (Newtonian), the problem (1.10) has a long history and was made famous by Poincaré in his 1902 treatise [44] on equilbrium figures. If the total angular momentum vanishes then the "problem of the equilibrium figure" reduces to mimimization of $-\mathcal{V}$ (with Newtonian $\alpha = n-2$) with its attractive gravitational potential. Poincaré claimed the unique solution was the ball. A complete solution to the problem had to wait almost a century with the work of Lieb [38] in 1977 whose proof was based on the strict Riesz rearrangement inequality.

The problem for equilibrium figures of anisotropic potentials as solutions of (1.10) remains an important open problem for general f. Unfortunately, for general f, the Riesz rearrangement inequality fails to hold true (cf. [46]). To our knowledge, even the criticality of the Wulff shape for (1.10) is unclear.

Outline of the article.
– In Section 2, we present some preliminary facts about the variational problem (1.2) and deduce some a priori structure and regularity properties of minimizers, provided they exist.

– In Section 3, we prove that (1.2) admits a minimizer when m is sufficiently small, while no minimizer exists when m is sufficiently large.

– In Section 4, we establish Theorem 1.1, first proving Theorem 1.3.

– In Section 5 , we prove Theorem 1.5.

– In Section 6, we address the inclusion of anisotropic potentials, proving Theorem 1.5.

– In Section 7, we conclude by noting and recalling some open problems.

–We finally include a few details in an appendix.

2. Preliminaries

Throughout the paper we denote constants that might change from line to line by C and keep track which parameters these constants depend on in parentheses. For specific constants that reappear elsewhere in the paper we use lower-case letters and denote their dependencies with subscripts such as $c_{n,f}$, $c_{n,\alpha}$.

2.1. **Basic properties of the surface energy.** The fact that $\mathcal{P}_f$ is uniquely minimized by translations of the *Wulff shape* K *of* f defined in (1.3) can be restated in a scaling invariant way via the Wulff inequality

$$\mathcal{P}_f(E) \geqslant n|E|^{(n-1)/n}|K|^{1/n}. \tag{2.1}$$

Throughout, we denote the dilation of the Wulff shape by $K_\rho = \rho K$. Recall that f_* denotes the Fenchel dual of f defined by

$$f_*(x) = \sup\{x \cdot \nu : f(\nu) \leqslant 1\},$$

with the Wulff shape K equivalently expressed as the unit ball for f_*. Let

$$\ell_f = \inf\{f(\nu) \,\big|\, \nu \in \mathbb{S}^{n-1}\}, \qquad L_f = \sup\{f(\nu) \,\big|\, \nu \in \mathbb{S}^{n-1}\}. \tag{2.2}$$

It follows that

$$\frac{1}{L_f} = \inf\{f_*(x) \,\big|\, x \in \mathbb{S}^{n-1}\}, \qquad \frac{1}{\ell_f} = \sup\{f_*(x) \,\big|\, x \in \mathbb{S}^{n-1}\}\,.$$

In particular, we observe that $B_{\ell_f} \subset K \subset B_{L_f}$, where B_r denotes the ball of radius r in $\mathbb{R}^n$.

2.2. **Basic properties of the nonlocal energy.** Fix $\alpha \in (0, n)$ and let $v_E : \mathbb{R}^n \to \mathbb{R}$ denote the Riesz potential of E given by

$$v_E(x) = \int_E \frac{dy}{|x-y|^\alpha}\,. \tag{2.3}$$

In this way, $\mathcal{V}(E) = \int_E v_E(x)\,dx$. Taking r such that $\omega_n r^n = |E|$, where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, we have

$$\|v_E\|_{L^\infty(\mathbb{R}^n)} \leqslant \|v_{B_r(0)}\|_{L^\infty(\mathbb{R}^n)} = v_{B_r(0)}(0) = \frac{n\omega_n}{n-\alpha} r^{n-\alpha}\,. \tag{2.4}$$

Furthermore, for $k = \lfloor n - \alpha \rfloor$ and $\beta \in (0,1)$ with $k + \beta < n - \alpha$, we have

$$\|v_E\|_{C^{k,\beta}(\mathbb{R}^n)} \leqslant C(n, |E|, k, \beta)\,; \tag{2.5}$$

see, e.g. [45, Lemma 3],[9, Proposition 2.1].

The functional $\mathcal{V}(E)$ is Lipschitz continuous with respect to the symmetric difference in the sense that there exists $c_{n,\alpha} > 0$ such that

$$|\mathcal{V}(E) - \mathcal{V}(F)| \leqslant c_{n,\alpha}\, m^{(n-\alpha)/n} |E \triangle F| \qquad \text{for } |E| \leqslant |F| = m. \tag{2.6}$$

Indeed, thanks to (2.4),

$$\begin{aligned}
\mathcal{V}(E) - \mathcal{V}(F) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_E(x)\chi_E(y) - \chi_F(x)\chi_F(y)}{|x-y|^\alpha}\, dxdy \\
&= \int_{\mathbb{R}^n} v_E(y)(\chi_E(y) - \chi_F(y))\, dy + \int_{\mathbb{R}^n} v_F(x)(\chi_E(x) - \chi_F(x))\, dx \\
&\leqslant \frac{2n\omega_n^{\alpha/n}}{n-\alpha} m^{(n-\alpha)/n} |E \triangle F|.
\end{aligned}$$

Hence, (2.6) holds with

$$c_{n,\alpha} := \frac{2n\omega_n^{\alpha/n}}{n-\alpha}\,. \tag{2.7}$$

2.3. **Scaling of the energy and initial energy bounds.** Given a set of finite perimeter $E \subset \mathbb{R}^n$, note that

$$\mathcal{E}_f(rE) = r^{n-1}\{\mathcal{P}_f(E) + r^{n+1-\alpha}\mathcal{V}(E)\}.$$

Hence, setting $\varepsilon = m^{(n+1-\alpha)/n}$, E_m is a minimizer of (1.2) with mass m if and only if $E = m^{-1/n}E_m$ is a minimizer of the variational problem

$$\inf\Big\{\mathcal{E}_{\varepsilon,f}(E) \;\Big|\; |E| = 1\Big\}, \quad \text{where} \quad \mathcal{E}_{\varepsilon,f}(E) := \mathcal{P}_f(E) + \varepsilon\,\mathcal{V}(E). \tag{2.8}$$

It will often be convenient to consider minimizers of this rescaled problem in place of (1.2). Let us give two initial bounds on the infimum in (2.8), which here and in the sequel we denote by $\bar{\mathcal{E}}_{\varepsilon,f}$. The first bound is essentially optimal for small ε, whereas the second is essentially optimal for large ε. First, comparing to K_r with $r = |K|^{-1/n} \leqslant 1/(\ell_f \omega_n^{1/n})$, we find that

$$\begin{aligned}
\bar{\mathcal{E}}_{\varepsilon,f} \leqslant \mathcal{E}_{\varepsilon,f}(K_r) &\leqslant n|K|r^{n-1} + C(n,\alpha)\varepsilon \\
&\leqslant C(n, \alpha, \ell_f, L_f),
\end{aligned} \tag{2.9}$$

where the third inequality holds only when $\varepsilon \leqslant 1$. Next, fix $N \in \mathbb{N}$, let $\rho = (N|K|)^{-1/n}$, and consider the sequence $\{E_k\}$ given by $E_k = \bigcup_{j=1}^N (K_\rho + kje_1)$, so that

$$\mathcal{E}_{\varepsilon,f}(E_k) = N^{1/n}|K|^{1/n} n + \varepsilon c_{n,\alpha} N^{(-n+\alpha)/n} + o_k(1).$$

Optimizing in N and letting $k \to \infty$, we find that

$$\bar{\mathcal{E}}_{\varepsilon,f} \leqslant C(n, \alpha, \ell_f, L_f)\, \varepsilon^{1/(n+1-\alpha)}\,. \tag{2.10}$$

2.4. **Quasi-minimality of minimizers.** Let us recall two useful notions of sets that almost minimize the surface energy $\mathcal{P}_f$. We say that E is a (Λ, r)-quasi-minimizer of $\mathcal{P}_f$ if for any $x \in \mathbb{R}^n$

$$\mathcal{P}_f(E) \leqslant \mathcal{P}_f(F) + \Lambda\,|E\triangle F| \qquad \text{for all } F \text{ with } \quad F\triangle E \subset\subset B_r(x)\,. \tag{2.11}$$

We say that E is a q-volume-constrained quasi-minimizer if

$$\mathcal{P}_f(E) \leqslant \mathcal{P}_f(F) + q\,|E\triangle F| \qquad \text{for all } F \text{ with } \quad |F| = |E|\,.$$

Since the potential $\mathcal{V}(E)$ is Lipschitz we can deduce that any minimizer of (2.8) satisfies both of these quasi-minimality properties:

Lemma 2.1. *Let E be a minimizer of (2.8). Then*

(i) *E is a $c_{n,\alpha}\varepsilon$-volume-constrained quasi-minimizer of $\mathcal{P}_f$, with $c_{n,\alpha} > 0$ given by (2.7), and*
(ii) *E is a $(\Lambda, 1)$-quasi-minimizer of $\mathcal{P}_f$ for some $\Lambda > 0$ which depends on f, n, α, and ε, and is bounded independently of ε for any $\varepsilon \leqslant 1$.*

The proof of Lemma 2.1 is standard, but we include it in Appendix A for the convenience of the reader. A classical argument, see for instance [42, Theorem 21.11], shows that (Λ, r_0)-quasi-minimizers satisfy uniform density estimates: provided $\Lambda r_0/\ell_f \leqslant 1$, if $x \in \partial E$ and $r < r_0$, then

$$c_0 \leqslant \frac{|E \cap B_r(x)|}{\omega_n r^n} \leqslant 1 - c_0 \qquad \text{with} \quad c_0 := \frac{\ell_f^n}{4^n L_f^n}. \tag{2.12}$$

Recall that a set E is *$\mathcal{P}_f$-indecomposable* if whenever $E = E_1 \cup E_2$ with E_1, E_2 disjoint and $\mathcal{P}_f(E) = \mathcal{P}_f(E_1) + \mathcal{P}_f(E_2)$, then $|E_1||E_2| = 0$. If a $\mathcal{P}_f$-indecomposable set E with $|E| \leqslant 1$ satisfies the lower density estimates of (2.12) up to scale r_0, then

$$\operatorname{diam} E \leqslant 2^{n+1}(c_0\omega_n)^{-1}\, r_0^{1-n}.$$

Indeed, let $d = \operatorname{diam} E$ and take $N = \lfloor d r_0^{-1} \rfloor$ points $\{x_i\}_{i=1}^N \subset \partial E$ such that $\{B_{r_0/2}(x_i)\}_{i=1}^N$ are pairwise disjoint. Then

$$1 = |E| \geqslant \sum_{i=1}^N |E \cap B_{r_0/2}(x_i)| \geqslant 2^{-n} N c_0 \omega_n r_0^n \geqslant 2^{-n-1} c_0 \omega_n d r_0^{n-1}.$$

Observe that minimizer E of (2.8) is indecomposable. Suppose $E = E_1 \cup E_2$ for disjoint sets E_1, E_2 with $\mathcal{P}_f(E) = \mathcal{P}_f(E_1) + \mathcal{P}_f(E_2)$. Applying the diameter bound to each $\mathcal{P}_f$ indecomposable component of E_1, E_2, we find that E_1 and $E_2 + ke_1$ are disjoint for k sufficiently large, so taking $G = E_1 \cup (E_2 + ke_1)$, we have $\mathcal{E}_{\varepsilon,f}(G) \leqslant \mathcal{E}_{\varepsilon,f}(E)$, with strict inequality unless $|E_1||E_2| = 0$.

Furthermore, note that for minimizers $E_\varepsilon \subset \mathbb{R}^n$ of the rescaled problem (2.8), using the estimate (2.6), we have that $\mathcal{P}_f(E_\varepsilon) \leqslant \mathcal{P}_f(K) + C(n,\alpha)\,\epsilon$. This implies that $E_\varepsilon \to K$ in $L^1(\mathbb{R}^n)$ (up to translation). Furthermore, the density estimates (2.12) paired with the L^1 convergence yield $d_H(\partial E_\varepsilon, \partial K) \to 0$ as $\varepsilon \to 0$, where d_H denotes the Hausdorff distance.

2.5. **Regularity of minimizers.** Suppose f is a smooth elliptic surface tension. The first variation of $\mathcal{P}_f(E)$ with respect to a variation generated by $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ is given by

$$\delta \mathcal{P}_f(E)[X] = \int_{\partial^* E} \operatorname{div}^{\partial^* E}\big(\nabla f \circ \nu_E\big) X \cdot \nu_E \, d\mathcal{H}^{n-1}$$

where $\operatorname{div}^{\partial^* E}$ denotes the tangential divergence along $\partial^* E$, and ∂^* denotes the reduced boundary. We define $H^f_E : \partial^* E \to \mathbb{R}$ by $H^f_E = \operatorname{div}^{\partial^* E}\big(\nabla f \circ \nu_E\big)$. Often H^f_E is called anisotropic mean curvature in analogy with the isotropic setting. The first variation of $\mathcal{V}(E)$ with respect to a variation generated by $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ is given by

$$\delta \mathcal{V}(E)[X] = \int_{\partial^* E} v_E(x) X \cdot \nu_E \, d\mathcal{H}^{n-1} \,,$$

with $v_E(x)$ as defined in (2.3).

We say that a set E is a critical point of (1.2) if $\delta(\mathcal{P}_f(E) + \mathcal{V}(E))[X] = 0$ for all variations with $\int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^{n-1} = 0$, i.e. variations that preserve volume to first order. Hence, a volume-constrained critical point E of (1.2) satisfies the Euler-Lagrange equation

$$H^f_E(x) + v_E(x) = \mu \qquad \text{for } x \in \partial^* E \,. \tag{2.13}$$

Here, μ is a Lagrange multiplier coming from the volume constraint.

We have the following regularity properties of minimizers of (1.2).

Theorem 2.2. *Fix $n \geqslant 2$ and $\alpha \in (0, n)$. Suppose f is a smooth elliptic surface tension and let E be a minimizer of (1.2) for mass m.*

(i) *The reduced boundary $\partial^* E$ is a $C^{2,\beta}$ hypersurface for all $\beta < \beta_0 := \min\{1, n-\alpha\}$.*

Furthermore, there exist m_4 and m_5 depending on n, f, and α with $m_4 \geqslant m_5$ such that the following statements hold.

(ii) *If $m \leqslant m_4$, then ∂E is a $C^{2,\beta}$ hypersurface for all $\beta \in (0, \beta_0)$, and in fact can locally be written as a small $C^{2,\beta}$ graph over the boundary of the Wulff shape of mass m.*

(iii) *If $m \leqslant m_5$, then E is uniformly convex.*

Theorem 2.2 can be deduced from known arguments and regularity results in the literature. We outline the proof in Appendix B for completeness, and following the proof we make several remarks about sharper forms of Theorem 2.2 that can be proven but are not needed in this paper.

Theorem 2.2 can be deduced from known arguments and regularity results for quasi-minimizers in the literature (see e.g. [2, 3, 8, 19]). Likewise sharper forms of Theorem 2.2 can be proven but are not needed in this paper. However, we remark on these extensions here for future reference.

3. Existence and nonexistence of minimizers

In this section we prove that the energy $\mathcal{E}_f$ admits a minimizer when m is sufficiently small, while no minimizer exists when m is sufficiently large.

Theorem 3.1. *Let $n \geqslant 2$, and let f be a surface tension with ℓ_f and L_f given by (2.2).*

(i) *For all $\alpha \in (0, n)$ there exists $m_1 = m_1(\alpha, n, \ell_f, L_f) > 0$ such that for all $m \leqslant m_1$, the variational problem (1.2) admits an essentially bounded and $\mathcal{P}_f$-indecomposable minimizer $E \subset \mathbb{R}^n$.*

(ii) *For all $\alpha \in (0,2)$, there exists $m_0 = m_0(\alpha, n, \ell_f, L_f)$ such that for all $m > m_0$, no minimizer exists in* (1.2).

It is not known whether minimizers of (1.2) exist for large masses, even in the isotropic case, for $\alpha \in [2, n)$. We prove Theorem 3.1(i) in the same way it was shown in the isotropic case in [36, Theorem 2.2] and [37, Theorem 3.1]; the details are given in Appendix C. The proof of Theorem 3.1(ii) combines an elegant argument of [27] with the diameter bound

$$\operatorname{diam} E \leqslant C(n, \alpha, \ell_f, L_f)\, m \tag{3.1}$$

for minimizer of (1.2) with mass m. This diameter estimate, shown in Appendix C, was originally established in [37, Lemma 7.2] by showing that the lower density estimates of (2.12) hold up to an improved scale.

Proof of Theorem 3.1(i). It is equivalent to show that there is $\varepsilon_1 = \varepsilon_1(n, f, \alpha) \leqslant 1$ such that a minimizer exists for the rescaled problem (2.8) for $\varepsilon \leqslant \varepsilon_1$. Let $\{F_k\}$ be a minimizing sequence for (2.8) with $|F_k| = 1$. Lemmas C.1 and C.2 show that there exists $\bar\rho$ such that for each k, there exists $\rho_k \leqslant \bar\rho$ satifsying

$$\mathcal{E}_{\varepsilon,f}(G_k) \leqslant \mathcal{E}_{\varepsilon,f}(F_k),$$

where G_k is a dilation of $F_k \cap B_{\rho_k}$ with $|G_k| = 1$ and $G_k \subset B_{R_0}(0)$ with R_0 depending on n, α, ℓ_f, and L_f. Such a sequence, having $\mathcal{P}_f(G_k) \leqslant C$, is compact in the L^1 topology, so up to a subsequence, $G_k \to E$ in L^1 with $|E| = 1$. The energy $\mathcal{E}_{\varepsilon,f}$ is lower semi-continuous with respect to L^1 convergence, so E is a minimizer of (2.8). The boundedness and indecomposability of a minimizer follow from the remarks in Section 2.4. □

Before proving Theorem 3.1(ii), let us fix some notation. For fixed $t \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$, we define the hyperplane

$$H_{\nu,t} = \{x \in \mathbb{R}^n \,\big|\, x \cdot \nu = t\}$$

and the corresponding half-spaces

$$H^+_{\nu,t} = \{x \in \mathbb{R}^n \,\big|\, x \cdot \nu > t\}, \qquad H^-_{\nu,t} = \{x \in \mathbb{R}^n \,\big|\, x \cdot \nu < t\}.$$

For a given set E, we let $E^\pm_{\nu,t} = E \cap H^\pm_{\nu,t}$.

Proof of Theorem 3.1(ii). Suppose E is a minimizer of (1.2) for mass m. For fixed $t \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$, arguing as we did to show indecomposability in Section 2.4, we have

$$\mathcal{E}_f(E) \leqslant \mathcal{E}_f(E^+_{\nu,t}) + \mathcal{E}_f(E^-_{\nu,t})\,. \tag{3.2}$$

Next, recall that for any set of finite perimeter E and $\nu \in \mathbb{S}^{n-1}$, we have

$$\mathcal{P}_f(E^+_{\nu,t}) + \mathcal{P}_f(E^-_{\nu,t}) = \mathcal{P}_f(E) + \{f(\nu) + f(-\nu)\}\mathcal{H}^{n-1}(E \cap H_{\nu,t})$$

for a.e. $t \in \mathbb{R}$ (see [42, Theorem 16.3 and Proposition 2.16]). So, rearranging (3.2) yields

$$\mathcal{V}(E) - \mathcal{V}(E^+_{\nu,t}) - \mathcal{V}(E^-_{\nu,t}) \leqslant \{f(\nu) + f(-\nu)\}\mathcal{H}^{n-1}(E \cap H_{\nu,t}) \tag{3.3}$$

for a.e. t. We will integrate both sides of (3.3) with respect to $t \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$. Integrating the right-hand side yields $2\mathcal{P}_f(B_1)m$. For the left-hand side, we observe that

$$\mathcal{V}(E) - \mathcal{V}(E^+_{\nu,t}) - \mathcal{V}(E^-_{\nu,t}) = 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\chi_{E^+_{\nu,t}}(x)\chi_{E^-_{\nu,t}}(y)}{|x-y|^\alpha}\,dxdy$$

and that

$$\chi_{E^+_{\nu,t}}(x)\chi_{E^-_{\nu,t}}(y) = \chi_{\{x\cdot\nu>t>y\cdot\nu\}}(x,y)\chi_E(x)\chi_E(y).$$

By the layer cake formula,

$$\int_{-\infty}^{\infty} \chi_{\{x\cdot\nu>t>y\cdot\nu\}}\,dt = \int_{-\infty}^{\infty} \chi_{\{(x-y)\cdot\nu>s>0\}}(x,y)\,ds = [(x-y)\cdot\nu]_+\,.$$

So, by Fubini's theorem, integrating the left-hand side of (3.3) with respect to $t\in\mathbb{R}$ gives us

$$2\int_E\int_E \frac{[(x-y)\cdot\nu]_+}{|x-y|^\alpha}\,dxdy \tag{3.4}$$

We now integrate (3.4) over $\nu\in\mathbb{S}^{n-1}$. Note that

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} [a\cdot\nu]_+\,d\mathcal{H}^{n-1} &= |a|\int_{\mathbb{S}^{n-1}} [a/|a|\cdot\nu]_+\,d\mathcal{H}^{n-1} = |a|\int_0^{\pi/2}\cos(\theta)\mathcal{H}^{n-2}(\mathbb{S}^{n-2}_\theta)\,d\theta\\ &= (n-1)\omega_{n-1}|a|\int_0^{\pi/2}\cos(\theta)\sin^{n-2}(\theta)\,d\theta = |a|\omega_{n-1}.\end{aligned}$$

So, again using Fubini's theorem, integrating both sides of (3.3) over $t\in\mathbb{R}$ and $\nu\in\mathbb{S}^{n-1}$ yields

$$2\omega_{n-1}\int_E\int_E \frac{1}{|x-y|^{\alpha-1}}\,dxdy \leqslant 2\mathcal{P}_f(B_1)m. \tag{3.5}$$

When $\alpha\in(0,1]$, the left-hand side is minimized by $B_{(m/\omega_n)^{1/n}}$ and hence is bounded below by $C(n,\alpha)\,m^{2-(\alpha-1)/n}$ for some constant $C(n,\alpha)>0$. It follows that the existence of a minimizer implies that

$$m\leqslant C(n,\alpha,\ell_f,L_f). \tag{3.6}$$

On the other hand, when $\alpha\in(1,2)$, the left-hand side of (3.5) is bounded below by $2\omega_{n-1}(\operatorname{diam}E)^{1-\alpha}m^2$. Rearranging this gives us the diameter lower bound

$$m^{1/(\alpha-1)}\leqslant C(n,\alpha,\ell_f,L_f)\operatorname{diam}E$$

Pairing this with (3.1), we establish (3.6) in this case as well. □

4. Smooth elliptic surface tensions: the Proof of Theorems 1.1 and 1.3

In this section, we establish Theorem 1.1. The main tool is Theorem 1.3 which states that, under suitable regularity assumptions, balls are the *only* bounded sets for which the first variation of the nonlocal energy $\mathcal{V}(E)$ is constant. We first prove Theorem 1.3 which, as described in Remark 1.4, which extends previous results for $0<\alpha<n-1$ to cases where the Riesz potential lacks C^1 regularity.

Proof of Theorem 1.3. Let us introduce some notation. For simplicity, we let $v=v_E$ throughout the proof. Following the notation in Section 3, for any $t\in\mathbb{R}$, let

$$H_t=\{x\cdot e_1=t\},\qquad H_t^-=\{x\cdot e_1<t\},\qquad H_t^+=\{x\cdot e_1>t\}.$$

Let

$$\Sigma_t = E\cap H_t^-$$

and, defining $x^t = (2t - x_1, x_2, \dots, x_n)$ to be the reflection of x across H_t, let

$$\Sigma_t' = \{x^t \,\big|\, x \in \Sigma_t\}.$$

Finally, let $v_t(x) = v(x^t)$ and set $G_t = E \setminus \overline{(\Sigma_t \cup \Sigma_t')}$. Observe that $G_t \subset H_t^+$.

Since E is bounded, Σ_t is empty for t sufficiently small and contains E for t sufficiently large. This means that

$$\lambda = \sup\{t \,\big|\, \Sigma_t' \subset E\}$$

is finite. Furthermore, we have either

Case 1. $\partial\Sigma_\lambda'$ is tangent to ∂E for some point $\bar{x} \in \partial E \setminus H_\lambda$, or
Case 2. H_λ is orthogonal to the tangent plane of ∂E at some point $\bar{x} \in \partial E \cap H_\lambda$.

We will show that in either case, G_λ is empty and thus E is symmetric across H_λ.

For any $x \in \mathbb{R}^n$ we may write

$$v(x) = \int_{\Sigma_\lambda} |x-y|^{-\alpha}\,dy + \int_{\Sigma_\lambda'} |x-y|^{-\alpha}\,dy + \int_{G_\lambda} |x-y|^{-\alpha}\,dy,$$
$$v_\lambda(x) = \int_{\Sigma_\lambda} |x-y|^{-\alpha}\,dy + \int_{\Sigma_\lambda'} |x-y|^{-\alpha}\,dy + \int_{G_\lambda} |x^\lambda-y|^{-\alpha}\,dy$$

and therefore we have

$$v_\lambda(x) - v(x) = \int_{G_\lambda} \left(|x^\lambda - y|^{-\alpha} - |x-y|^{-\alpha} \right) dy. \tag{4.1}$$

In particular, for any $x \in H_\lambda^-$, the right-hand side is strictly positive unless G_λ is empty.

Let us first suppose that Case 1 occurs. Since $\bar{x} \in \partial\Sigma_\lambda' \cap \partial E$ and $\bar{x} \notin H_\lambda$, it follows from construction that $\bar{x}$ is the reflection x^λ for some $x \in \partial E \cap \partial\Sigma_\lambda$. Since x and $\bar{x}^\lambda$ both lie in ∂E, we have $v(\bar{x}) = v_\lambda(\bar{x}) = \text{const}$. Hence (4.1) implies that G_λ is empty and so E is symmetric about H_λ.

Next, suppose that Case 2 occurs. Note that e_1 is parallel to the tangent plane of ∂E at $\bar{x}$. Hence, up to a translation and a rotation of E that fixes its orientation with respect to e_1, we may assume that $\bar{x} = 0$ and the tangent plane to ∂E at zero is the plane $\{x_n = 0\}$. Moreover, we may locally express ∂E as the graph over its tangent plane. More specifically, letting

$$x = (x', x_n) \quad \text{and} \quad B_r'(0) = \{|x'| < r\} \subset \mathbb{R}^{n-1},$$

for r suitably small, we may find a $C^{k,\gamma}$ function $\varphi : B_r'(0) \to \mathbb{R}$:

$$\partial E \cap B_r(0) = \left\{ \big(x', \varphi(x')\big) : x' \in B_r'(0) \right\}. \tag{4.2}$$

If $\alpha \in (0, n-1)$, we can argue exactly as in [39]. In this case, v is differentiable (recall (2.5)) and thus $\partial_{e_1} v = 0$. In particular, for $h > 0$ small, we have

$$|v(\bar{x} + he_1) - v(\bar{x} - he_1)| = o(h). \tag{4.3}$$

Now suppose that $\alpha \in [n-1, n-1/3)$. For $h > 0$ small, let $\varphi_h = \varphi(-he_1)$ and consider the sequence $\{x_h\} \subset \partial E \cap H_\lambda^-$ defined by

$$x_h = -he_1 + \varphi_h e_n.$$

In this way, $x_h^\lambda = he_1 + \varphi_h e_n \in H_\lambda^+$. We claim that

$$|v(x_h^\lambda) - v(x_h)| \leqslant Ch^{1+\eta} \tag{4.4}$$

for some $\eta > 0$. Indeed, recall from (2.5) that for any $\beta < \min\{n-\alpha, 1\}$, we have $\|v\|_{C^{0,\beta}} \leqslant C$. So, as $v(x) = \bar{c}$ for some constant $\bar{c} \in \mathbb{R}$ for all $x \in \partial E$,

$$|v(x_h^\lambda) - v(x_h)| = |v(x_h^\lambda) - \bar{c}| \leqslant C\mathrm{dist}\big(x_h^\lambda, \partial E\big)^\beta.$$

In order to estimate the right-hand side, we argue separately when $\alpha \in [n-1, n-1/2)$ and when $\alpha \in [n-1/2, n-1/3)$.

First, suppose that $\alpha \in [n-1, n-1/2)$. In this case, by the hypotheses, $\partial E \in C^{1,\gamma}$ for $1+\gamma > 1/(n-\alpha)$. Consider a Taylor expansion of φ: since $\varphi(0) = |\nabla\varphi(0)| = 0$, we have

$$\varphi(x') = O\big(|x'|^{1+\gamma}\big).$$

Let $y_h = he_1 + \varphi(he_1)e_n \in \partial E$ by (4.2). Hence,

$$\begin{aligned}\mathrm{dist}\big(x_h^\lambda, \partial E\big) \leqslant \big|x_h^\lambda - y_h\big| &= |\varphi(-he_1) - \varphi(he_1)| \\ &\leqslant |\varphi(-he_1)| + |\varphi(he_1)| \leqslant Ch^{1+\gamma}.\end{aligned}$$

Since $1+\gamma > 1/(n-\alpha)$, we may take $\beta < n-\alpha$ large enough such that $(1+\gamma)\beta > 1$. This yields (4.4) in this case.

Next, suppose that $\alpha \in [n-1/2, n-1/3)$. Then, by assumption $\partial E \in C^{2,\gamma}$ with $2+\gamma > 1/(n-\alpha)$. Now, we use the fact that in the e_1 direction, ∂E separates from its reflection as distance to the power $2+\gamma$. Indeed, a Taylor expansion of φ in this case yields

$$\varphi(x') = D^2\varphi(0)[x', x'] + O\big(|x'|^{2+\gamma}\big).$$

In this way, again letting $y_h = he_1 + \varphi(he_1)e_n$, we see that

$$\mathrm{dist}\big(x_h^\lambda, \partial E\big) \leqslant \big|x_h^\lambda - y_h\big| = |\varphi(he_1) - \varphi(-he_1)| \leqslant Ch^{2+\gamma}.$$

Note that in general ∂E separates only quadratically from its reflection across H_λ; the key point here is that we have chosen our sequence so the x' components of the sequence and its reflection correspond to the reflection $x' = -x'$. Again, since $2+\gamma > 1/(n-\alpha)$, we may choose $\beta < n-\alpha$ large enough so that $(2+\gamma)\beta > 1$, proving (4.4).

Next, we claim that if G_λ is nonempty, we have

$$|v(x_h^\lambda) - v(x_h)| \geqslant Ch\,. \tag{4.5}$$

Indeed, for fixed h, (4.1) implies that

$$v_\lambda(x_h) - v(x_h) = \int_{G_\lambda} \big|x_h^\lambda - y\big|^{-\alpha} - \big|x_h - y\big|^{-\alpha}\, dy \tag{4.6}$$

and the integrand is positive. In order to establish (4.5), consider the strip $S_\delta = \{x \,\big|\, |x \cdot e_1| < \delta\}$. Since G_λ is an open subset of H_λ^+, taking $\delta > 0$ sufficiently small, we may find some open ball B that is compactly contained in $G_\lambda \setminus S_\delta$. Choosing h sufficiently small, we see that for any $y \in B$, the function $f(t) = |te_1 + \varphi_h e_n - y|^{-\alpha}$ is smooth for $t \in (-h, h)$. Thus, we apply the mean value theorem to the function $f(t)$ on this interval in order to rewrite the integrand on the right-hand side of(4.6) as

$$|x_h^\lambda - y|^{-\alpha} - |x_h - y|^{-\alpha} = 2h(-\alpha|\tilde{x}_h - y|^{-\alpha-2}(\tilde{x}_h - y) \cdot e_1) \tag{4.7}$$

for some $\tilde{x}_h = te_1 + \varphi_h e_n$ depending on y with $t \in (-h, h)$. Notice further that $(y - x_h^\lambda) \cdot e_1 \geqslant \delta/2$ (and hence $(y - \tilde{x}_h) \cdot e_1 \geqslant \delta/2$) for all $y \in B$, provided h is taken to be small enough. Therefore, for $y \in B$ we have

$$v_\lambda(x_h) - v(x_h) \geqslant -2h\alpha \int_B |\tilde{x}_h - y|^{-\alpha-2}(\tilde{x}_h - y) \cdot e_1 \, dy \geqslant Ch.$$

This establishes (4.5). Note that when $\alpha \in (0, n-1)$, we can repeat the argument above to obtain that

$$|v(\bar{x} + he_1) - v(\bar{x} - he_1)| \geqslant Ch \tag{4.8}$$

provided G_λ is nonempty.

Now, letting h tend to zero, we see that (4.3) (resp. (4.4)) and (4.8) (resp. (4.5)) are in opposition, and so we deduce that G_λ is empty.

For both cases 1 and 2, we have deduced that E is symmetric about H_λ. Since e_1 was chosen arbitrarily, we find that E is a ball. □

We now show that Theorem 1.1 follows directly from Theorem 1.3.

Proof of Theorem 1.1. We first prove statement (i). Let K_r be the dilation of the Wulff shape K with $|K_r| = m$, that is, take $r = (m/|K|)^{1/n}$. Direct computation shows that $H^f_{K_r}(x) = (n-1)/r$ for all $x \in \partial K_r$. Hence, recalling (2.13), if K_r is a critical point of (1.2), then

$$v_{K_r}(x) = \mu' \qquad \text{for } x \in \partial K_r$$

where $\mu' = \mu - (n-1)/r$. Since K_r is smooth, applying Theorem 1.3 concludes the proof of (i).

The proof of (ii) similarly follows from Theorem 1.3. To this end, take $\bar{m}_1$ to be equal to the constant m_4 from Theorem 2.2. Then, if E is a minimizer of (1.2) for mass $m \leqslant \bar{m}_1$, then E is of class $C^{2,\beta}$ for all $\beta < n - \alpha$ and E satisfies the Euler-Lagrange equation (2.13) with $\mu = \mu_E$. Then for any $r > 0$, setting $F = rE$, we have

$$H^f_{rE}(y) + v_{rE}(y) = r^{-1} H^f_E(y/r) + r^{n-\alpha} v_E(y/r) \qquad \forall y \in \partial F\,.$$

Setting $x = y/r \in \partial E$, this implies

$$r^{-1} H^f_E(x) + r^{n-\alpha} v_E(x) = r^{-1}\mu_E + (r^{n-\alpha} - r^{-1}) v_E(x) \qquad \forall x \in \partial E\,.$$

So, if F is a critical point of (1.2) for $m = |F|$, then the left-hand side is equal to a constant μ_F. Rearranging, this yields

$$v_E(x) = \frac{\mu_F - r^{-1}\mu_E}{r^{n-\alpha} - r^{-1}} = \text{constant}.$$

For $\alpha \in (0, n - 1/2)$, we may readily apply Theorem 1.3 to conclude that E is a ball. For $\alpha \geqslant n - 1/2$, in order to apply Theorem 1.3 to E, we verify that we may find β satisfying $2 + \beta > 1/(n-\alpha)$ and $\beta < n - \alpha$. Our assumption that $\alpha < n - (\sqrt{2} - 1)$ ensures that this is possible.

Finally arguing as above, we conclude that $H^f_E(x)$ is constant on the boundary of this ball E. However, the Alexandrov-type theorem proven in [34] asserts that the only smooth set with constant H^f_E is the Wulff shape. We conclude that the Wulff shape of f is a ball, and thus f is a multiple of the Euclidean norm. □

5. Crystalline surface tensions: the Proof of Theorem 1.2

In this section, we prove Theorem 1.2, providing an example of a surface tension f for which the Wulff shape *is* the minimizer of (1.2) for sufficiently small mass. When f fails to be C^1 or elliptic, one can no longer compute the first variation of $\mathcal{P}_f$ with respect to arbitrary smooth compactly supported variations. A key point becomes understanding among what variations one can compute the Euler-Lagrange equation. As pointed out in the introduction, this is a subtle point and the focus of considerable research in crystalline mean curvature flow. Here we rely on the following 2D structure result from [22, Theorem 7]:

Theorem 5.1 (Figalli-Maggi)**.** *Let $n = 2$ and let f be a crystalline surface tension, so that the Wulff shape is a convex polygon with outer unit normals $\{\nu_i\}_{i=1}^N$. Then there exists a positive constant q_0 such that if E is a q-volume-constrained quasi-minimizer with $q < q_0$, then E is a convex polygon with $\nu_E \in \{\nu_i\}_{i=1}^N$ for $\mathcal{H}^1$-a.e. $x \in \partial E$.*

This result is in some sense related to that of [51] where Taylor considers motion of polygonal curves by crystalline curvature and proves that line segments flow in their normal directions, keeping their same normals. Therefore the segments expand and contract to maintain their directions and adjacencies as they flow towards a steady state.

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Let E be a minimizer of the rescaled problem (2.8) with the surface tension f as in (1.4). It is equivalent to show that there exists ε' such that the Wulff shape is the global minimizer of (2.8) for $\varepsilon \leqslant \varepsilon'$. Recall from Lemma 2.1 that a minimizer E of (2.8) is a q-volume-constrained quasi-minimizer of $\mathcal{P}_f$ with $q = \varepsilon c_{n,\alpha}$, so the hypotheses of the Theorem 5.1 hold for ε' sufficiently small. Furthermore, since E satisfies the density estimates (2.12), a classical argument shows that $d_H(\partial E, \partial K) \to 0$ as $\varepsilon \to 0$, where d_H is the Hausdorff distance. Hence, we deduce further that, up to decreasing ε', we have $\{\nu_E\} = \{\pm e_1, \pm e_2\}$ for $\mathcal{H}^1$-a.e. $x \in \partial E$. That is, up to a translation, E is a rectangle of the form

$$R_a = [-a/2, a/2] \times [-1/2a, 1/2a]$$

for some $a \in (1-\omega(\varepsilon), 1+\omega(\varepsilon))$, where $\omega(\varepsilon)$ is a (nonexplicit) modulus of continuity with $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. In this way, we need only to consider the one dimensional family of variations (with respect to a) : E is a minimizer of (2.8) if and only if it is a minimizer of the one-dimensional variational problem

$$\inf \left\{ \mathcal{E}_{\varepsilon,f}(R_a) : R_a = [-a/2, a/2] \times [-1/2a, 1/2a] \right\}.$$

Note that $f(\pm e_1) = f(\pm e_2) = 1/2$. Hence, the energy of R_a is

$$\begin{aligned} \mathcal{E}_{\varepsilon,f}(R_a) &= \mathcal{P}_f(R_a) + \varepsilon \mathcal{V}(R_a) \\ &= \left(a + \frac{1}{a}\right) + \varepsilon \int_{-a/2}^{a/2} \int_{-1/(2a)}^{1/(2a)} \int_{-a/2}^{a/2} \int_{-1/(2a)}^{1/(2a)} \frac{dx_1 dy_1 dx_2 dy_2}{\left((x_1 - x_2)^2 + (y_1 - y_2)^2\right)^{\alpha/2}}. \end{aligned}$$

Note that

$$\frac{d}{da}\mathcal{P}_f(R_a) = 1 - \frac{1}{a^2}, \qquad \frac{d^2}{da^2}\mathcal{P}_f(R_a) = \frac{2}{a^3}. \tag{5.1}$$

Using the change of variables $\widetilde{x}_i = x_i/a$ and $\widetilde{y}_i = ay_i$ for $i = 1, 2$, we rewrite the nonlocal term as

$$\mathcal{V}(R_a) = \iiiint_{[-1/2,1/2]^4} \left(a^2(\widetilde{x}_1 - \widetilde{x}_2)^2 + a^{-2}(\widetilde{y}_1 - \widetilde{y}_2)^2\right)^{-\alpha/2} d\widetilde{x}_1 d\widetilde{y}_1 d\widetilde{x}_2 d\widetilde{y}_2 \,.$$

Differentiating with respect to a we see that

$$\frac{d}{da}\mathcal{V}(R_a) = \iiiint_{[-1/2,1/2]^4} \left[\frac{-\alpha}{2}\left(a^2(\widetilde{x}_1 - \widetilde{x}_2)^2 + a^{-2}(\widetilde{y}_1 - \widetilde{y}_2)^2\right)^{-\alpha/2-1} \right.$$
$$\left. \left(2a(\widetilde{x}_1 - \widetilde{x}_2)^2 - \frac{2}{a^3}(\widetilde{y}_1 - \widetilde{y}_2)^2\right) \right] d\widetilde{x}_1 d\widetilde{y}_1 d\widetilde{x}_2 d\widetilde{y}_2 \,;$$

hence, $(d/da)\Big|_{a=1} \mathcal{V}(R_a) = 0$. Pairing this with (5.1) we see that K is a critical point of $\mathcal{E}_{\varepsilon,f}(R_a)$. Furthermore, direct computation shows that

$$\frac{d^2}{da^2}\Big|_{a=1} \mathcal{E}_{\varepsilon,f}(R_a) = 2 - C(\alpha)\varepsilon \geqslant \frac{1}{2}$$

for sufficiently small $\varepsilon > 0$; hence, K is a stable (in the sense of second variations) critical point of the energy functional $\mathcal{E}_{\varepsilon,f}$ for sufficiently small $\varepsilon > 0$. Now using this stability property we will proceed as in [1, Theorem 5.1] and [9, Theorem 2.10] to show that, up to translations, the Wulff shape K is the unique global minimizer.

Suppose there exists a sequence $\{E_k\}$ of global minimizers of (2.8) with $\varepsilon_k \to 0$. Note that, up to translations, $E_k = R_{a_k}$ with $a_k \in (1 - \omega(\varepsilon_k), 1 + \omega(\varepsilon_k))$. In particular, $a_k \to 1$ and $E_k \to K$ in L^1. Now, since K is a strictly stable critical point of $\mathcal{E}_{\varepsilon_k,f}$, for $k > k_0$,

$$\mathcal{E}_{\varepsilon_k,f}(R_{a_k}) \geqslant \mathcal{E}_{\varepsilon_k,f}(K) + \frac{1}{4}(a_k - 1)^2 + o\big((a_k - 1)^2\big) > \mathcal{E}_{\varepsilon_k,f}(K) + \frac{1}{8}(a_k - 1)^2 \,.$$

We conclude that $a_k = 1$ for k sufficiently large, concluding the proof. □

Remark 5.2 (Higher dimensions). In their recent work Figalli and Zhang extend the rigidity result of Theorem 5.1 to higher dimensions (see [23, Theorem 1.1]). Using this as our starting point, we can argue as in the proof above, and reduce the admissible class of variations to a family defined via a finite number of parameter. To be precise, given

$$f(\nu) = \frac{1}{2}\|\nu\|_{\ell^1(\mathbb{R}^n)} = \frac{1}{2}\left(\sum_{i=1}^n |\nu \cdot e_i|\right)$$

E is a minimizer of (2.8) if and only if it is a minimizer of the $(n-1)$-dimensional variational problem

$$\inf \Bigg\{ \mathcal{E}_{\varepsilon,f}(R_{a_1,a_2,\ldots,a_{n-1}}) :$$
$$R_{a_1,a_2,\ldots,a_{n-1}} = \left[-\frac{a_1}{2}, \frac{a_1}{2}\right] \times \left[-\frac{a_2}{2}, \frac{a_2}{2}\right] \times \cdots \times \left[-\frac{1}{2a_1 \cdots a_{n-1}}, \frac{1}{2a_1 \cdots a_{n-1}}\right] \Bigg\}.$$

A calculation of the first and second derivatives with respect to the parameters $a_1, \dots, a_{n-1}$ shows that the Wulff shape $R_{1,\dots,1}$ is a stable critical point of the energy $\mathcal{E}_{\varepsilon,f}$ for ε sufficiently small, and we can conclude its minimality as above.

6. Fully anisotropic models: the Proof of Theorem 1.5

In this section we prove Theorem 1.5. Before we proceed, let us remark that the problems (1.5) and (1.6) admit minimizers when m is sufficiently small. Indeed, existence for (1.6) follows from the direct method in the calculus of variations, while existence for (1.5) follows by arguing as in Lemmas C.1, C.2, and Theorem 3.1(i), thanks to the subcritical scaling of $\mathcal{U}_i$ with respect to $\mathcal{P}_f$.

The interesting feature of these three classes of problems is that each $\mathcal{U}_i(E)$ defined in (1.7) is maximized by the Wulff shape among sets of a fixed volume, so the variational problems (1.5) and (1.6) exhibit the same type of competition between terms as in the classical liquid drop model (1.1). Let us see that $\mathcal{U}_1(E)$ is maximized by K among sets E with $|E| = |K|$. Indeed, note that the supremum in $\mathcal{U}_1(K)$ is attained at $y = 0$; up to a translation we may assume the same for E. Then

$$\mathcal{U}_1(K) - \mathcal{U}_1(E) = \int_{K\setminus E} f_*(x)^{-\alpha}\,dx - \int_{E\setminus K} f_*(x)^{-\alpha}\,dx > |K \setminus E| - |E \setminus K| = 0 \tag{6.1}$$

since $f_*(x) > 1$ in $E \setminus K$ and $f_*(x) < 1$ in $K \setminus E$. The computation is analogous for $\mathcal{U}_2$ and $\mathcal{U}_3$.

A key tool in the proof of Theorem 1.5 is the following strong form of the quantitative anisotropic isoperimetric inequality which was shown in [43, Proposition 1.4]:

Proposition 6.1. *Let f be a smooth elliptic surface tension with corresponding surface energy $\mathcal{P}_f$ and Wulff shape K. Let E be a set such that $|E| = |K|$ and $\mathrm{bar}E = \mathrm{bar}K$, where $\mathrm{bar}E = |E|^{-1}\int_E x\,dx$ denotes the barycenter of E. Suppose*

$$\partial E = \{x + u(x)\nu_K(x) \,\big|\, x \in \partial K\}$$

where $u : \partial K \to \mathbb{R}$ is in $C^1(\partial K)$. There exist C and μ depending on n and f such that if $\|u\|_{C^1(\partial K)} \leqslant \mu$, then

$$C\|u\|^2_{H^1(\partial K)} \leqslant \mathcal{P}_f(E) - \mathcal{P}_f(K). \tag{6.2}$$

The idea of the proof of Theorem 1.5 the following: Using the Wulff shape as a competitor in (1.5) and applying Proposition 6.1, we find that $\mathcal{U}_i(K) - \mathcal{U}_i(E)$ is bounded below by a constant (increasing with m^{-1}) multiple of the squared distance between E and K. On the other hand, the criticality and C^2 bound of $\mathcal{U}_i(K)$ ensure that this gap is at most a fixed constant multiple of the squared distance. These ideas are in the spirit of those used in [21, 37] in the setting of (1.1).

Proof of Theorem 1.5. Let us start by observing the scaling of the energies:

$$\begin{aligned}
\mathcal{U}_1(rE) &= r^{\gamma_1}\mathcal{U}_1(E) && \gamma_1 = n - \alpha\,,\\
\mathcal{U}_2(rE) &= r^{\gamma_2}\mathcal{U}_2(E) && \gamma_2 = n + \beta\,,\\
\mathcal{U}_3(rE) &= r^{\gamma_3}\mathcal{U}_3(E) + r^n \log(r)|E| && \gamma_3 = n\,.
\end{aligned}$$

Thus in place of (1.5) and (1.6) respectively, we set $\varepsilon_i = (m|K|^{-1})^{(\gamma_i - n+1)/n}$ and study the equivalent variational problems

$$\inf\Big\{\mathcal{P}_f(E) + \varepsilon_1 \mathcal{U}_1(E) \,\Big|\, |E| = |K|\Big\} \tag{6.3}$$

and

$$\inf\Big\{\mathcal{P}_f(E) + \varepsilon_i \mathcal{U}_i(E) \,\Big|\, |E| = |K|,\ E \subset B_{c_{n,f}}\Big\} \qquad i = 2, 3\,. \tag{6.4}$$

To prove Theorem 1.5, it is equivalent to show that for $i = 1, 2, 3$, there exists $\varepsilon_i' = \varepsilon_i'(n, f, \mathcal{U}_i)$ such that for $\varepsilon \leqslant \varepsilon_i'$, the only minimizers of (6.3) and (6.4) are Wulff shapes.

Claim: For $i = 1, 2, 3$, suppose E_i is a global minimizer of (6.3) or (6.4). Then, up to a translation and provided ε_i is sufficiently small, ∂E_i is a small C^1 graph over ∂K:

$$\partial E_i = \{x + u_i(x)\nu_K(x) \,|\, x \in \partial K\},$$

where $u_i : \partial K \to \mathbb{R}$ has $\|u_i\|_{C^1(\partial K)} \leqslant \mu/3$ with μ as in Proposition 6.1.

Let us assume the claim for now, since the proof differs between (6.3) and (6.4). Up to replacing $\mu/3$ by μ, we may further assume that $\mathrm{bar}E_i = \mathrm{bar}K$. We can therefore apply Proposition 6.1 to find that

$$C\|u_i\|^2_{H^1(\partial K)} \leqslant \mathcal{P}_f(E_i) - \mathcal{P}_f(K) \tag{6.5}$$

with the constant $C > 0$ given in (6.2). Now, taking the Wulff shape as a competitor in (6.3) and (6.4) and rearranging the energy, we find that

$$\mathcal{P}_f(E_i) - \mathcal{P}_f(K) \leqslant \varepsilon_i\big(\mathcal{U}_i(K) - \mathcal{U}_i(E_i)\big).$$

Letting $X_i = u_i\nu_K$, a Taylor expansion of $\mathcal{U}_i(E_i)$ yields

$$\mathcal{U}_i(E_i) = \mathcal{U}_i(K) + \delta\mathcal{U}_i(K)[X] + \frac{1}{2}\delta^2\mathcal{U}_i(K)[X, X] + \mu\, O(\|u\|^2_{H^1(\partial K)})\,. \tag{6.6}$$

Here $\delta\mathcal{U}_i(K)$ and $\delta^2\mathcal{U}_i(K)$ denote the first and second variations respectively. Direct computation (recalling that $f_*(x) = 1$ for $x \in \partial K$) shows that the first variations are given by

$$\delta\mathcal{U}_1(K)[X_1] = \int_{\partial K} u_1\, d\mathcal{H}^{n-1}\,, \quad \delta\mathcal{U}_2(K)[X_2] = \int_{\partial K} u_2\, d\mathcal{H}^{n-1}\,, \quad \delta\mathcal{U}_3(K)[X_3] = 0\,. \tag{6.7}$$

Arguing as in [43, proof of Lemma 4.1], the fact that $|E_i| = |K|$ implies that

$$\int_{\partial K} u_i\, d\mathcal{H}^{n-1} = -\frac{1}{2}\int_{\partial K} H_K u_i^2\, d\mathcal{H}^{n-1} + \mu\, O(\|u_i\|^2_{H^1(\partial K)}) \tag{6.8}$$

where H_K is the (isotropic) mean curvature of K. Next, a somewhat lengthy yet standard computation (making use of the identity $\nabla f_*(x) = \nu_K(x)/f(\nu_K(x))$ for $x \in \partial K$) shows

that

$$\begin{aligned}
\delta^2\mathcal{U}_1(K)[X_1,X_1] &\geqslant \int_{\partial K} u_1^2 H_K + u_1 \nabla u_1\cdot\nu_K - \alpha u_1^2 f(\nu_K)^{-1}\,d\mathcal{H}^{n-1}\\
&\geqslant -\kappa_1\|u_1\|^2_{H^1(\partial K)}\,,\\
\delta^2\mathcal{U}_2(K)[X_2,X_2] &\geqslant \int_{\partial K} u_2^2 H_K + u_2 \nabla u_2\cdot\nu_K + \beta u_2^2 f(\nu_K)^{-1}\,d\mathcal{H}^{n-1}\\
&\geqslant -\kappa_2\|u_2\|^2_{H^1(\partial K)}\,,\\
\delta^2\mathcal{U}_3(K)[X_3,X_3] &\geqslant \int_{\partial K} u_3^2 f(\nu_K)^{-1}\,d\mathcal{H}^{n-1} \geqslant -\kappa_3\|u_3\|^2_{H^1(\partial K)}
\end{aligned} \tag{6.9}$$

for some constants $\kappa_i > 0$.

Together, (6.6), (6.7), (6.8), and (6.9) imply that

$$\mathcal{U}_i(K) - \mathcal{U}_i(E_i) \leqslant C_i\|u_i\|^2_{H^1(\partial K)}. \tag{6.10}$$

where $C_i = C(n, f, \mathcal{U}_i)$. Finally, combining (6.5) and (6.10), we see that taking ε_i sufficiently small forces $u_i = 0$, and hence E_i is the Wulff shape.

Let us now prove the claim, arguing separately for $\mathcal{U}_1(E)$ and for $\mathcal{U}_2(E)$, $\mathcal{U}_3(E)$. Let $y_E \in \mathbb{R}^n$ be a point attaining the supremum in $\mathcal{U}_1(E)$. Then, using (6.1), for any E, F with $|E| = |F|$ we have

$$\mathcal{U}_1(E) - \mathcal{U}_1(F) \leqslant \int_{E\triangle F} f_*(x - y_E)^{-\alpha}\,dx \leqslant \int_{rK} f_*(x)^{-\alpha}\,dx$$

where r is chosen so that $|rK| = |E\triangle F|$. The term on the right-hand side is equal to $\frac{n}{n-\alpha}|K|r^{n-\alpha}$. In this way, arguing as in the proof of Lemma 2.1, we find that a minimizer E_1 of (1.5) satisfies

$$\mathcal{P}_f(E_1) \leqslant \mathcal{P}_f(F) + \Lambda\,|E_1\triangle F|^{\gamma_1/n} \qquad \text{whenever } E_1\triangle F \subset\subset B_r(x)$$

for some $x \in R^n$ and $r \leqslant \ell_f/L_f$. = Since $\gamma_1 > n-1$, the results of [47] show that $\partial^* E$ is locally $C^{1,\eta}$ for $\eta = \frac{1-\alpha}{2}$. Then, using the first variation of the energy given by

$$H^f_{E_1}(x) + \varepsilon_1\, f_*(x)^{-\alpha} = \text{const} \qquad \forall x \in \partial^* E_1,$$

we can repeat Steps 2 and 3 of the proof of Theorem 2.2 to conclude the claim in this case.

Now, for any $E, F \subset B_{c_{n,f}}$, we have

$$\mathcal{U}_2(E) - \mathcal{U}_2(F) \leqslant \int_{E\triangle F} f_*(x - y_E)^{\beta}\,dx \leqslant (2c_{n,f}/\ell_f)^{\beta}|E\triangle F|.$$

Similarly,

$$\mathcal{U}_3(E) - \mathcal{U}_3(F) \leqslant \int_{E\triangle F} \log(f_*(x - y_F))\,dx \leqslant \log(2c_{n,f}/\ell_f)|E\triangle F|.$$

As such, we may argue as in Lemma 2.1 to find that if E_i is a minimizer of (6.4) for $i = 2, 3$, then E_i is a quasi-minimizer of the surface energy. Again, using the Euler-Lagrange equations

$$H^f_{E_2}(x) - \varepsilon_2\, f_*(x)^{\beta} = \text{const} \qquad \forall x \in \partial^* E_2,$$

$$H^f_{E_3}(x) - \varepsilon_3 \log f_*(x) = \text{const} \qquad \forall x \in \partial^* E_3,$$

we may repeat Steps 2 and 3 of the proof of Theorem 2.2 to conclude the claim. □

7. Open Problems

We conclude by mentioning and recalling some important open problems. First note that all the open problems for the liquid drop problem (1.1) carry over to (1.2). In the case of smooth anisotropies, it is not clear what more one could hope to prove about minimizers of (1.2), with a clear characterization unlikely. Here, numerical computations could prove quite insightful in a qualitative assessment of the difference between the shape of minimizers and the Wulff shape. For crystalline anisotropies there remains much to be done in determining the minimality of the Wulff shape. The most tractable problem would be to generalize Theorem 1.2 to all regular polygons in 2D.

We would like to end by highlighting the open problem alluded to in Remark 1.6. This problem is, in our opinion, a rather fundamental problem (1.10) which pertains to the "equilibrium figure" with an anisotropic potential. Unfortunately, given that Riesz rearrangement techniques fail, it is not clear what techniques one could employ.

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Appendix A. Proof of Lemma 2.1

Let us prove Lemma 2.1.

Proof. Taking F to be any competitor with $|E| = |F| = 1$, (2.6) implies that

$$\mathcal{P}_f(E) \leqslant \mathcal{P}_f(F) + \varepsilon\big(\mathcal{V}(E) - \mathcal{V}(F)\big) \leqslant \mathcal{P}_f(F) + c_{n,\alpha}\varepsilon\,|E \triangle F|.$$

This proves (i). To show (ii), we employ the typical trick of showing that E is a minimizer of the unconstrained variational problem

$$\inf\left\{\mathcal{E}_{\varepsilon,f}(F) + Q\big||F| - 1\big| \,\middle|\, F \subset \mathbb{R}^n\right\}. \tag{A.1}$$

for some fixed constant $Q = 5n\bar{\mathcal{E}}_{\varepsilon,f}$, where we adopt the shorthand $\bar{\mathcal{E}}_{\varepsilon,f} = \mathcal{E}_{\varepsilon,f}(E)$. It suffices to show that if

$$\mathcal{E}_{\varepsilon,f}(F) + Q\big||F| - 1\big| \leqslant \bar{\mathcal{E}}_{\varepsilon,f}, \tag{A.2}$$

then $|F| = 1$. For any F satisfying (A.2), let $G = rF$ so that $|G| = 1$. We note immediately that $r \in [1, 2)$. Indeed,

$$\begin{aligned} \mathcal{E}_{\varepsilon,f}(G) < \mathcal{E}_{\varepsilon,f}(F) &\leqslant \bar{\mathcal{E}}_{\varepsilon,f} && \text{if } r < 1, \\ Q/2 \leqslant Q(1 - r^{-n}) = Q||F| - 1| &\leqslant \bar{\mathcal{E}}_{\varepsilon,f} && \text{if } r \geqslant 2, \end{aligned}$$

violating the minimality of E in (2.8) and our choice of Q respectively. Now, since $n - 1 < 2n - \alpha$, we have

$$\mathcal{E}_{\varepsilon,f}(F) \geqslant r^{\alpha-2n}\mathcal{E}_{\varepsilon,f}(G) \geqslant r^{\alpha-2n}\bar{\mathcal{E}}_{\varepsilon,f}$$

and so rearranging (A.2) gives

$$Q(1-r^{-n}) \leqslant \bar{\mathcal{E}}_{\varepsilon,f}(1-r^{\alpha-2n})\,.$$

By concavity, we have the bounds $1-r^{-n} \geqslant (r-1)/2$ and $1-r^{\alpha-2n} \leqslant 2n(r-1)$ for $r \in [1,2)$. Thus,

$$Q(r-1) \leqslant 4n\bar{\mathcal{E}}_{\varepsilon,f}(r-1),$$

forcing $r=1$ by our choice of Q. We conclude that E is a minimizer of (A.1).

Hence, taking any F with $|E\triangle F| \leqslant 1$ (and hence $|F| \leqslant 2$) as a competitor in (A.1) and recalling (2.6), we obtain

$$\begin{aligned}\mathcal{P}_f(E) &\leqslant \mathcal{P}_f(F) + \varepsilon\left(\mathcal{V}(F)-\mathcal{V}(E)\right) + Q\big||F|-1\big| \\ &\leqslant \mathcal{P}_f(F) + (\varepsilon\, c_{n,\alpha}+Q)|E\triangle F|\end{aligned}$$

Thus, E is a $(\Lambda,1)$-quasi-minimizer with $\Lambda = \varepsilon\, c_{n,\alpha}+Q$. If $\varepsilon \leqslant 1$, then Λ can be taken to be independent of ε. □

Appendix B. Regularity of minimizers

Next we outline the proof of the regularity result.

Proof of Theorem 2.2. As before, it will be convenient to consider (2.8) in place of the equivalent problem (1.2); by rescaling, the same statements will hold for minimizers of (1.2). We also assume without loss of generality that we have multiplied f by a constant so that the Wulff shape K has unit mass.

Step 1: Quasi-minimality and $C^{1,\gamma}$ regularity. Let E be a minimizer of (2.8). Then, by Lemma 2.1, E is a q-volume-constrained quasi-minimizer of $\mathcal{P}_f$. The epsilon-regularity theory for quasi-minimizers of $\mathcal{P}_f$, see [2, 3, 8, 19], ensures that $\partial^* E$ is of class $C^{1,\gamma}$ for $\gamma \in (0,1)$. To state this more precisely, let us introduce a bit of notation. For $x \in \mathbb{R}^n$, $r>0$, and $\nu \in \mathbb{S}^{n-1}$, we define

$$\begin{aligned}\mathbf{C}_\nu(x,r) &= \{y \in \mathbb{R}^n \,\big|\, |p_\nu(y-x)| < r,\ |q_\nu(y-x) < r\}, \\ \mathbf{D}_\nu(x,r) &= \{y \in \mathbb{R}^n \,\big|\, |p_\nu(y-x)| < r,\ |q_\nu(y-x)| = 0\},\end{aligned}$$

where $q_\nu(y) = y\cdot\nu$ and $p_\nu(y) = y-(y\cdot\nu)y$. We then define the *cylindrical excess* of E at x in direction ν at scale r to be

$$\operatorname{exc}(E,x,r,\nu) = \frac{1}{r^{n-1}}\int_{\mathbf{C}_\nu(x,r)\cap\partial^* E} \frac{|\nu_E-\nu|^2}{2}\, d\mathcal{H}^{n-1}\,.$$

For all $\gamma \in (0,1)$ there exist constants $C(n,f,\gamma)$ and δ depending on n, f and γ such that if a quasi-minimizer E satisfies

$$\operatorname{exc}(E,x,r,\nu) + q\, r < \delta \tag{B.1}$$

then there exists $u \in C^{1,\gamma}(\mathbf{D}_\nu(x,r))$ with $u(x)=0$ such that

$$\begin{aligned}\mathbf{C}_\nu(x,r/2)\cap\partial^* E &= (\mathrm{Id}+u\nu)(\mathbf{D}_\nu(x,r/2)), \\ \|u\|_{C^0(\mathbf{D}_\nu(x,r/2))} &< C(n,f,\gamma)\, r \operatorname{exc}(E,x,r,\nu)^{1/(2n-2)}, \\ \|\nabla u\|_{C^0(\mathbf{D}_\nu(x,r/2))} &< C(n,f,\gamma) \operatorname{exc}(E,x,r,\nu)^{1/(2n-2)}, \\ r^\gamma [\nabla u]_{C^{0,\gamma}(\mathbf{D}_\nu(x,r/2))} &< C(n,f,\gamma) \operatorname{exc}(E,x,r,\nu)^{1/2}.\end{aligned} \tag{B.2}$$

For any $x \in \partial^* E$, (B.1) will be satisfied at sufficiently small scale r, and we conclude that $\partial^* E$ is locally a $C^{1,\gamma}$ hypersurface.

Step 2: $C^{2,\beta}$ regularity. Furthermore, as we noted in equation (2.13), E satisfies the Euler-Lagrange equation

$$H^f_E(x) + \varepsilon\, v_E(x) = \mu \qquad \forall x \in \partial^* E$$

in a distributional sense. Given $x \in \partial^* E$, if we choose r to be suitably small so that (B.1) holds, the Euler-Lagrange equation reads

$$\mathrm{div}'(\nabla' f'(\nabla' u(z)) = \varepsilon v_E(z, u(z)) - \mu \qquad \forall z \in \mathbf{D}_\nu(x, r/2). \tag{B.3}$$

Here, $f'(z) = f(-z, 1)$ and div' and ∇' indicate derivatives with respect to the z variable. Applying Schauder estimates (see [31, Theorem 6.2]),

$$\|u\|_{C^{2,\beta}(\mathbf{D}_\nu(x,r/4))} \leqslant C(r, f, \|u\|_{C^{1,\gamma}(\mathbf{D}_\nu(x,r/2)}, \beta)\|v_E - \mu\|_{C^{0,\beta}(\mathbf{D}_\nu(x,r/2))}\,. \tag{B.4}$$

Finally, (2.5) shows that $\|v_E\|_{C^{0,\beta}} \leqslant C(n, \alpha, \beta, |E|)$ for all $\beta \in (0, \min\{1, n-\alpha\})$. This concludes (i).

Step 3: Improved convergence. To establish (ii) and (iii), consider a sequence of minimizers E_ε of (2.8) with $\varepsilon \to 0$. Provided $\varepsilon \leqslant 1$, r_0 and q in (2.11) can be taken to be independent of ε, and so E_ε satisfy uniform (in ε) density estimates. Thanks to the diameter bound (3.1) and the Wulff inequality (2.1), $E_\varepsilon \to K$ in L^1, and thanks to the density estimates, $d_H(\partial E_\varepsilon, \partial K) \to 0$, where d_H is the Hausdorff distance. We argue as in [17] to obtain a uniform graphicality scale r on which:

$$\begin{aligned} \partial K \cap \mathbf{C}_\nu(x, r/2) &= (\mathrm{Id} + u\nu)(\mathbf{D}_\nu(x, r/2)), \\ \partial E_\varepsilon \cap \mathbf{C}_\nu(x, r/2) &= (\mathrm{Id} + u_\varepsilon\nu)(\mathbf{D}_\nu(x, r/2)), \\ \|u_\varepsilon - u\|_{L^\infty} &\to 0. \end{aligned} \tag{B.5}$$

and the estimates of (B.2) hold uniformly in ε. The key point here is the continuity of the cylindrical excess with respect to L^1 convergence of quasi-minimizers with uniform r_0 and q; since ∂K is smooth, E_ε will satisfy the flatness assumption (B.1) for every $x \in \partial E_\varepsilon$ and at a uniform scale in ε.

Finally, since

$$\mu_\varepsilon = \frac{1}{n}\Big((n-1)\mathcal{P}_f(E_\varepsilon) + \varepsilon(2n-\alpha)\mathcal{V}(E_\varepsilon)\Big) \leqslant C(\mathcal{P}_f(K) + \mathcal{V}(K)),$$

we see that the right-hand side of (B.4) is bounded by a constant independent of ε. By the Arzelà-Ascoli theorem, $\|u_\varepsilon - u\|_{C^{2,\beta'}} \to 0$ for all $\beta' \leqslant \beta$. In particular, for ε sufficiently small, this yields (ii). Since K is uniformly convex, it follows that ∂E_ε is uniformly convex as well provided ε is sufficiently small. □

Remark B.1 (Higher regularity). Of course, starting from the $C^{1,\gamma}$ regularity coming from quasi-minimality, the regularity of $\partial^* E$ (and then ∂E for small masses) can be improved much as the Euler-Lagrange equation will allow. Indeed, in place of Theorem 2.2(i), one can prove that the reduced boundary $\partial^* E$ is a $C^{2+k,\beta}$ hypersurface for all $\beta \in (0,1)$ and $k + \beta < n - \alpha$. To establish higher regularity, one differentiates (B.3) and applies the same Schauder estimates (B.4) to derivatives of u, making use of the smoothness of f and (2.5).

Remark B.2 (Lower regularity assumptions on f). For convenience we have assumed that $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ throughout the paper. Provided $f \in C^{2,\gamma}(\mathbb{R}^n \setminus \{0\})$, one can still establish Theorem 2.2(i) for $\beta < \min\{\gamma, n-\alpha\}$. The proof follows the one given above verbatim. In fact, to establish the $C^{1,\gamma}$ regularity of $\partial^* E$, we only need f to be elliptic with $f \in C^{1,1}(\mathbb{R}^n \setminus \{0\})$, this follows from the results of [20].

Remark B.3 (Quantitative estimates). In the case $\alpha \in (0, n-1)$, one can adapt ideas from [22] to make parts (ii) and (iii) of Theorem 2.2 quantitative in terms of the mass: there is a critical mass $m_6 = m_6(n, f, \alpha, \beta_0)$ and a constant $C = C(n, f, \alpha, \beta_0)$ such that under the hypotheses of Theorem 2.2 with $m \leqslant m_6$, and setting $F = (|K|/m)^{1/n} E$,

$$\max_{\partial F} |\nabla^2 f(\nu_F) \nabla \nu_F - \mathrm{Id}_{T_x \partial F}| \leqslant C m^{2\beta/(n+2\beta)}.$$

Such quantitative estimates were shown in [22, Theorem 2] for a related class of problems; while our nonlocal repulsion term $\mathcal{V}(E)$ does not fall into the class of potential terms studied there, their proof extends to our setting for $\alpha \in (0, n-1)$ with only minor adjustments. The only notable difference comes in the study of the second variation, where one must bound an additional term in the second variation of $\mathcal{V}(E)$ that does not appear for the potentials studied in [22]. In the case $\alpha \in [n-1, n)$, there are some subtle integrability issues for the second variation of $\mathcal{V}(E)$ and we do not know if the estimates can be made quantitative in this case.

Appendix C. Auxiliary lemmas toward existence and diameter bound

The following "non-optimality criterion" of [37], which follows by direct comparison and the Wulff inequality, is key in establishing both existence and the diameter bound.

Lemma C.1. *Fix $n \geqslant 2$, $\varepsilon > 0$ and a surface tension f with ℓ_f and L_f given by (2.2). There exists $\delta_0 = \delta_0(n, \alpha, \ell_f, L_f)$ such that the following holds. Suppose $|F| = 1$ and $\mathcal{E}_{\varepsilon,f}(F) \leqslant 2\bar{\mathcal{E}}_{\varepsilon,f}$. If $F = F_1 \cup F_2$ for two nonempty disjoint sets F_1, F_2 with*

$$\mathcal{P}_f(F_1) + \mathcal{P}_f(F_2) - \mathcal{P}_f(F) \leqslant \mathcal{P}_f(F_2)/2 \tag{C.1}$$

and

$$|F_2| \leqslant \delta_0 \min\{1, \varepsilon^{-n/(n+1-\alpha)}\}, \tag{C.2}$$

then $\mathcal{E}_{\varepsilon,f}(\hat{F}_1) < \mathcal{E}_{\varepsilon,f}(F)$, where $\hat{F}_1 = rF_1$ such that $|\hat{F}_1| = 1$.

Proof. Let $\delta = |F_2| \in (0,1)$, so that $\hat{F} = (1-\delta)^{1/n} F_1$. Observe that (C.1) implies that $\mathcal{E}_{\varepsilon,f}(F_1) \leqslant \mathcal{E}_{\varepsilon,f}(F) - \mathcal{P}_f(F_2)/2$. Hence, by (C.1),

$$\begin{aligned}
\mathcal{E}_{\varepsilon,f}(\hat{F}) &\leqslant (1+\delta)^{(2n-\alpha)/n} \mathcal{E}_{\varepsilon,f}(F_1) \\
&\leqslant (1 + C(n,\alpha)\,\delta) \mathcal{E}_{\varepsilon,f}(F_1) \\
&\leqslant (1 + C(n,\alpha)\,\delta) \Big(\mathcal{E}_{\varepsilon,f}(F) - \mathcal{P}_f(F_2)/2 \Big).
\end{aligned}$$

So, thanks to the Wulff inequality (2.1), we find that $\mathcal{E}_{\varepsilon,f}(\hat{F}) - \mathcal{E}_{\varepsilon,f}(F)$ is bounded above by

$$C(n,\alpha)\,\delta \bar{\mathcal{E}}_{\varepsilon,f} - \delta^{(n-1)/n} n |K|^{1/n},$$

which, recalling (2.9) and (2.10), is strictly negative provided $\delta < \delta_0$ for some $\delta_0 > 0$ sufficiently small. □

Lemma C.2. *Fix $n \geqslant 2$ and a surface tension f with ℓ_f and L_f given by (2.2). There exists $\varepsilon_1(n, \alpha, \ell_f, L_f) \leqslant 1$ and ρ_0 such that the following holds. Let F be a set of finite perimeter with $|F| = 1$ and $\mathcal{E}_{\varepsilon,f}(F) \leqslant \bar{\mathcal{E}}_{\varepsilon,f} + \varepsilon$. Then, for some $\rho \in [L_f/(\ell_f \omega_n^{1/n}), \rho_0)$ and after a translation, the sets*

$$F_1 = F \cap B_\rho \qquad \text{and} \qquad F_2 = F \setminus B_\rho$$

satisfy (C.1) *and* (C.2).

Proof. Let F be a set of finite perimeter with $|F| = 1$ and $\mathcal{E}_{\varepsilon,f}(F) \leqslant \bar{\mathcal{E}}_{\varepsilon,f} + \varepsilon$. Set $s = |K|^{-1/n}$, so that $|K_s| = 1$, and replace F with a translation $F + x_0$ such that $|(F + x_0) \triangle K_s| = \inf_x |(F + x) \triangle K_s|$. Note that $K_s \subset B_\rho$ provided $\rho \geqslant L_f/(\ell_f \omega_n^{1/n})$. For all such ρ, set

$$F_1^\rho = F \cap B_\rho \qquad \text{and} \qquad F_2^\rho = F \setminus B_\rho.$$

We claim that there exists a constant $\rho_0 = \rho_0(n, \ell_f, L_f)$ such that (C.1) and (C.2) are satisfied for some $\bar{\rho} \leqslant \rho_0$ provided ε_1 is sufficiently small. Let us first see that (C.2) is satisfied for every $\rho \geqslant L_f/(\ell_f \omega_n^{1/n})$, provided ε_1 is small enough. Indeed, note that

$$\mathcal{P}_f(F) - \mathcal{P}_f(K_s) \leqslant \varepsilon(\mathcal{V}(K_s) - \mathcal{V}(F)) + \varepsilon \leqslant C(n, f, \alpha)\, \varepsilon\,.$$

Then, since $F_2^\rho \subset E \triangle K_s$ for every , we have $|F_2^\rho| \leqslant |E \triangle K_s| \to 0$ as $\mathcal{P}_f(F) - \mathcal{P}_f(K_s) \to 0$. So, for any such ρ, we have that $|F_2^\rho|$ satisfies (C.2) for ε_1 sufficiently small.

Now, let $\bar{\rho}$ be the smallest constant greater than or equal to $L_f/(\ell_f \omega_n^{1/n})$ such that (C.1) is satisfied. For a.e. $\rho > 0$, we have

$$\text{(C.3)}\quad \mathcal{P}_f(F_1^\rho) + \mathcal{P}_f(F_2^\rho) - \mathcal{P}_f(F) = \int_{\partial B_\rho \cap F} f(\nu_{B_\rho}) + f(-\nu_{B_\rho})\, d\mathcal{H}^{n-1} \leqslant 2 L_f \mathcal{H}^{n-1}(\partial B_\rho \cap F)\,;$$

see [42, Theorem 16.3 and Proposition 2.16]. Thus, by the definition of $\bar{\rho}$, for a.e. $\rho \in [L_f, \bar{\rho})$ we have

$$\mathcal{H}^{n-1}(\partial B_\rho \cap F) \geqslant \mathcal{P}_f(F_2^\rho)/4L_f.$$

Define the function $U(\rho) = |F \setminus B_\rho| = |F_2^\rho|$. Then for a.e. $\rho \in [L_f, \bar{\rho})$,

$$U'(\rho) = -\mathcal{H}^{n-1}(\partial B_\rho \cap F) \leqslant -\mathcal{P}_f(F_2^\rho)/4L_f \leqslant -c_1 |F_2^\rho|^{(n-1)/n} = -c_1 U(\rho)^{(n-1)/n}\,,$$

where $c_1 := n|K|^{1/n}/4L_f$. Integrating from L_f to $\bar{\rho}$ and noting that $U(L_f) \leqslant C(n, L_f)$, we find that $\bar{\rho} \leqslant \rho_0(n, \ell_f, L_f)$. This concludes the proof. □

Let us now prove the diameter bound given in (3.1).

Proof of (3.1). It is equivalent to show that a minimizer of (2.8) satisfies

$$\operatorname{diam} E \leqslant C(n, \alpha, \ell_f, L_f) \varepsilon^{(n-1)/(n+1-\alpha)}.$$

Thanks to Section 2.4, it suffices to show that for any $x \in E$, the lower density estimate of (2.12) holds for all $x \in E$ up to scale $\bar{r} \geqslant c_1\, \varepsilon^{-1/(n+1-\alpha)}$ with the constant c_1 as in the previous proof.

Fix $x \in E$ and let $F_1^r = E \setminus B_r(x)$ and $F_2^r = E \cap B_r(x)$. Let $\bar{r}$ be the smallest $r > 0$ such that

$$\text{(C.4)}\qquad \mathcal{P}_f(F_1^r) + \mathcal{P}_f(F_2^r) - \mathcal{P}_f(E) \leqslant \mathcal{P}_f(F_2^r)/2\,.$$

Note that $\bar r > \omega_n^{-1/n}\delta_0\varepsilon^{-1/(n+1-\alpha)}$, otherwise we may apply Lemma C.1 to contradict the minimality of E. Note also that $F_1^{\bar r}$ is nonempty. Indeed, if not, then $E \subset B_{\bar r}(x)$ (up to a null set), and in particular $\bar r \geqslant \omega_n^{-1/n}$. In this case, we have

$$\mathcal{E}_{\varepsilon,f}(E) \geqslant \varepsilon \mathcal{V}(E) \geqslant \varepsilon 2^{-\alpha}\omega_n^{\alpha/n},$$

contradicting (2.10).

Now, (C.4) allows us to extend the usual proof of lower density estimates of (2.12) up to scale $\bar r$. Indeed, for a.e. $r > 0$, we have

$$\mathcal{P}_f(F_1^r) + \mathcal{P}_f(F_2^r) - \mathcal{P}_f(E) = \int_{\partial B_r \cap E} f(\nu_{B_r}) + f(-\nu_{B_r})\, d\mathcal{H}^{n-1} \leqslant 2L_f \mathcal{H}^{n-1}(\partial B_r \cap E)\,,$$

and for any $r < \bar r$, the left-hand side is bounded above by $\mathcal{P}_f(F_2^r)/2$, so

$$\mathcal{H}^{n-1}(\partial B_r \cap E) \geqslant \mathcal{P}_f(F_2^r)/4L_f$$

for a.e. $r \in (0, \bar r)$. Hence, setting $U(r) = |E \cap B_r| = |F_2^r|$, we have for a.e. $r \in (0, \bar r)$

$$U'(r) = \mathcal{H}^{n-1}(\partial B_r \cap E) \geqslant \mathcal{P}_f(F_2^r)/4L_f \geqslant c_1 |F_2^r|^{(n-1)/n} = c_1 U(r)^{(n-1)/n}\,,$$

where $c_1 = n|K|^{1/n}/4L_f \geqslant n\omega_n^{1/n}\ell_f/4L_f$. Integrating from 0 to r for any $r \leqslant \bar r$ completes the proof. □

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Department of Mathematics and Statistics, McGill University, Montréal, QC
E-mail address: rustum.choksi@mcgill.ca

School of Mathematics, Institute for Advanced Study, Princeton, NJ
E-mail address: neumayer@ias.edu

Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA
E-mail address: iatopaloglu@vcu.edu