

# NONLOCAL SHAPE OPTIMIZATION VIA INTERACTIONS OF ATTRACTIVE AND REPULSIVE POTENTIALS

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ABSTRACT. We consider a class of nonlocal shape optimization problems for sets of fixed mass where the energy functional is given by an attractive/repulsive interaction potential in power-law form. We find that the existence of minimizers of this shape optimization problem depends crucially on the value of the mass. Our results include existence theorems for large mass and nonexistence theorems for small mass in the class where the attractive part of the potential is quadratic. In particular, for the case where the repulsion is given by the Newtonian potential, we prove that there is a critical value for the mass, above which balls are the unique minimizers, and below which minimizers fail to exist. The proofs rely on a relaxation of the variational problem to bounded densities, and recent progress on nonlocal obstacle problems.

## 1. INTRODUCTION

In this note we address the following nonlocal shape optimization problem:

$$(P) \quad \text{Minimize} \quad E(\Omega) := \int_{\Omega} \int_{\Omega} K(x-y) dx dy$$

over measurable sets  $\Omega \subset \mathbb{R}^d$  ( $d \geq 2$ ) of finite measure  $|\Omega| = m$ .

Here  $K : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  is a locally integrable, lower semicontinuous, radial function, and  $|\Omega|$  denotes the Lebesgue measure of the set  $\Omega$ . In particular, we are interested in interaction potentials in the power-law form

$$(1.1) \quad K(x) := \frac{|x|^q}{q} - \frac{|x|^p}{p}$$

where  $-d < p < q$  with  $p, q \neq 0$ .

These sums of attractive and repulsive power-law potentials have collective effect which is repulsive at short ranges but attractive at long ranges (see Figure 1). We will focus on positive attraction  $q > 0$  and Riesz potential repulsions  $-d < p < 0$ ; the majority of our results pertain to quadratic attraction  $q = 2$ , and some require  $p$  to be at or below  $2 - d$ , the exponent of the Newtonian potential. Our results are valid in any dimension  $d \geq 2$  with the understanding that when  $d = 2$  the Newtonian repulsion (corresponding to  $p = 2 - d = 0$ ) is given by  $\log|x|$ , i.e., the kernel (1.1) is  $K(x) = (1/q)|x|^q - \log(|x|)$  when  $p = 2 - d$ . Moreover, when  $p = 2 - d$  the repulsive part of the energy is determined by the  $H^{-1}$ -norm of the characteristic function and is equal to  $\|\chi_{\Omega}\|_{H^{-1}}^2$  up to a constant. We use the notation  $\chi_{\Omega}$  for the characteristic (indicator) function of a set.

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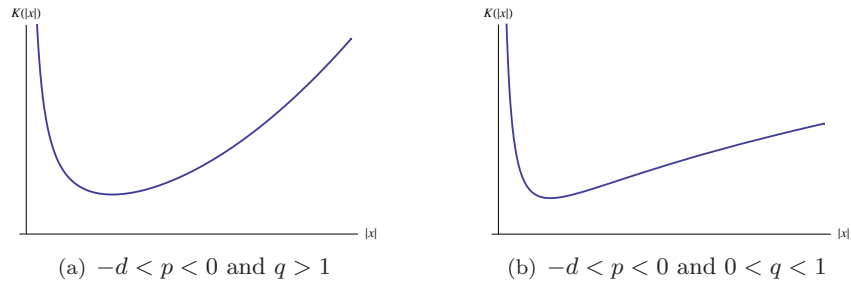


FIGURE 1. Generic examples of  $K$  for various values of  $q$  and  $p$ .

The problem **(P)** is a toy example of shape optimization problems where repulsive interactions at short distances compete with attraction at long distances. As far as we know this is the first work to address such problems. It is closely related to the problem of minimizing the nonlocal interaction energy

$$(1.2) \quad \mathbb{E}(\rho) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x-y) \rho(x) \rho(y) dx dy$$

over non-negative densities  $\rho \in L^1(\mathbb{R}^d)$  of given mass  $\|\rho\|_{L^1(\mathbb{R}^d)} = m$ . Such functionals appear in a class of well-studied self-assembly/aggregation models (e.g. see [4, 18, 22, 23] and the references therein). Under broad assumptions on the kernels, the existence of global minimizers [10, 11, 13, 33] and qualitative properties of local minimizers [2, 12] of these energies along with convex approximations of minimizers via analytical [17] and numerical [3] techniques have recently been investigated. These results do not directly extend to **(P)**, because a sequence of densities given by the indicator functions of measurable sets may converge weakly to densities taking values strictly between zero and one. Nevertheless, we are able to exploit the relation between the two problems to obtain existence and non-existence results for **(P)**.

The purpose of this study is to lay out the foundations for addressing **(P)**, focusing mostly on the case of quadratic attraction. In particular, we prove:

**Theorem 1.1.** *Let  $K$  be of the form (1.1), and let  $m > 0$ .*

- (i) *For  $q = 2$  and  $-d < p \leq 2 - d$  and for sufficiently small  $m$ , the problem **(P)** does not have a solution.*
- (ii) *For  $q = 2$  and  $-d < p < 0$  and for sufficiently large  $m$ , the ball of volume  $m$  is the unique solution of **(P)** up to translations.*
- (iii) *For  $q = 2$  and  $p = 2 - d$ , the unique solution of **(P)** is a ball of volume  $m$  if  $m \geq \omega_d$ , where  $\omega_d$  denotes the volume of the unit ball in  $\mathbb{R}^d$ . If  $m < \omega_d$ , the problem **(P)** does not have a solution.*

Our approach to Theorem 1.1 is via a *relaxation* of **(P)** wherein the energy (1.2) is minimized over densities  $\rho$  with  $0 \leq \rho \leq 1$  almost everywhere. We will denote this relaxed problem by **(RP)** and note that existence of minimizers was recently established in [13]. In Section 4, we show that **(P)** has a solution if and only if the relaxed problem has a solution which is a characteristic function (Theorem 4.4). We also derive the first variation of **(RP)** and show that local minimizers are compactly supported. These results hold for general kernels. In Section 5 we turn our attention to power-law potentials and consider the quadratic attraction case. After

establishing the uniqueness of minimizers we first prove part (i) of Theorem 1.1 via a recent regularity result of Carrillo, Delgadino and Mellet [12] for local minimizers of  $E$  over probability measures where they prove the connection with solutions of certain nonlocal obstacle problems and utilize their regularity [9, 32]. Then we show that balls satisfy the first-order variational inequalities corresponding to  $(\mathbf{RP})$  when the mass is sufficiently large and prove parts (ii) and (iii) of Theorem 1.1. Our results exploit the special nature (convexity) of the energy  $E$  for  $q = 2$ . We believe the basic approach to their proof should extend to all  $q > 0$ . We address the challenges of such extensions in Section 6 and also mention when we can expect minimizers that are not necessarily balls.

Our conclusions and the consideration of  $(\mathbf{P})$  are motivated by a number of old and new shape optimization problems which we now describe in the physically most relevant case of three dimensions.

## 2. RELATED SHAPE OPTIMIZATION PROBLEMS

We start with a problem of Poincaré on the shape of a fluid [31]. Assuming vanishing total angular momentum, the total potential energy in a fluid body, represented by a set  $\Omega \subset \mathbb{R}^3$ , is given by

$$- \int_{\Omega} \int_{\Omega} \frac{C}{|x-y|} dx dy,$$

where  $-C|x-y|^{-1}$  is the Newtonian potential resulting from the gravitational attraction between two points  $x$  and  $y$  in the fluid, and  $C > 0$  is a physical constant. After rescaling, *Poincaré's variational problem* is given by

$$\begin{cases} \text{Minimize} & - \int_{\Omega} \int_{\Omega} \frac{1}{|x-y|} dx dy \\ \text{over measurable sets } \Omega \subset \mathbb{R}^3 & \text{with } |\Omega| = m. \end{cases}$$

Poincaré asserted that among all shapes with fixed mass, the unique shape of lowest energy is a ball, and proved this statement for sufficiently smooth sets. He referred to previous work of Lyapunov but was critical of its incompleteness. It was not until almost a century later that all the details were sorted out by Lieb [27] wherein the heart of the matter lies in the rearrangement ideas of Steiner for the isoperimetric inequality. These ideas are captured in the Riesz rearrangement inequality and its development (cf. [7, 28]). On the other hand, the maximum energy is not attained, as by breaking up the shape and spreading out one can drive the energy to 0.

Another classical variational problem with similar conclusions is the *isoperimetric problem*:

$$\begin{cases} \text{Minimize} & \text{perimeter}(\Omega) \\ \text{over sets } \Omega \subset \mathbb{R}^3 & \text{of finite perimeter with } |\Omega| = m. \end{cases}$$

It is of course well-known that the only minimizers are balls. Again, the maximum does not exist.

The energies in both these problems are purely attractive in that they share an, albeit different, incentive for set elements to stay together. When these are placed in direct opposition by subtracting the energies, one obtains the *nonlocal isoperimetric problem*, which stated in dimension  $d = 3$  is

$$(\mathbf{NLIP}) \quad \begin{cases} \text{Minimize} & \text{perimeter}(\Omega) + \int_{\Omega} \int_{\Omega} \frac{1}{|x-y|} dx dy \\ \text{over sets } \Omega \subset \mathbb{R}^3 & \text{of finite perimeter with } |\Omega| = m. \end{cases}$$

Here, the Newton potential  $|x - y|^{-1}$  represents the electrostatic repulsion between two points  $x$  and  $y$ , and the double integral represents the Coulomb energy of a uniform charge distribution on  $\Omega$ . The two terms are now in direct competition: Balls are *best* (minimizers) for the first term but *worst* (maximizers) for the second. This functional was first introduced in [14, 15] in studying the small volume fraction asymptotics of the Ohta-Kawasaki functional. It was conjectured that there exists a critical mass  $m_c$  such that minimizers are balls for  $m \leq m_c$  and fail to exist otherwise. There has recently been much work on the **(NLIP)** (see e.g. [5, 20, 24–26, 29, 30]). To date what is known is that there exist two constants  $m_1 \leq m_2$  such that

- (i) balls are the unique minimizers if  $m \leq m_1$ , and
- (ii) minimizers fail to exist if  $m > m_2$ .

It remains open whether or not  $m_1 = m_2$ . Thus the heuristic picture emerges that the perimeter completely dominates up to a critical mass, beyond which the Coulomb repulsion is strong enough to break sets apart.

In the **(NLIP)** the attraction, that is the incentive for the set to remain together, is via perimeter, a local quantity involving derivatives, while the repulsion results from a pairwise interaction potential. As such the short and long-range interactions are inherently different<sup>1</sup>. It is thus natural to consider problems where both attraction and repulsion are dictated by pairwise interaction potentials in power-law form, for example, minimizers of

$$(2.1) \quad \frac{1}{2} \int_{\Omega} \int_{\Omega} |x - y|^2 dx dy + \int_{\Omega} \int_{\Omega} \frac{1}{|x - y|} dx dy$$

over sets  $\Omega \subset \mathbb{R}^3$  with  $|\Omega| = m$ . This is the special case of **(P)** with  $q = 2$ ,  $p = -1$  and  $d = 3$ . It can be viewed as toy problem for the total potential energy of *spring-like* media which at short distances experience Coulombic repulsion and at longer distances experience the usual Hookean attraction. As in the **(NLIP)**, balls are best for the first term but worst for the second. However the role of the mass  $m$  is reversed according to the different scaling of the attractive and repulsive terms in (2.1) with repulsion dominating for small  $m$  and attraction dominating for large  $m$ . While in the **(NLIP)** the lack of existence of minimizers is due to mass escaping to infinity, here it is due to oscillations. Moreover, unlike for the **(NLIP)**, here we can explicitly identify the critical threshold below which minimizers fail to exist and above which the unique minimizer is a ball.

In this short paper we make a first step at addressing existence vs. nonexistence for the general problem **(P)**, depending on the mass parameter  $m$ . Here there is a surprising lack of general mathematical tools: For controlling the attractive part of the interaction potential, there is nothing like the well-developed regularity theory for minimal surfaces, which greatly benefited the analysis of both the local and nonlocal isoperimetric problems, and recently, the analysis of variational problems with nonlocal derivatives. On the other hand, the Riesz rearrangement inequality which was the key to solving Poincaré's problem, goes in the wrong direction.

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<sup>1</sup>Recently there has also been a significant interest in nonlocal set interactions via nonlocal derivatives (see e.g. [1, 8, 19] and in particular [34] for a review). Here the repulsion is of Riesz-type and the attraction is created by the interaction of a set  $\Omega$  with its complement  $\Omega^c$ . Specifically, the nonlocal energy considered in these works is given by

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\chi_{\Omega}(x) - \chi_{\Omega}(y))^2}{|x - y|^{d+s}} dx dy$$

for some  $\Omega \subset \mathbb{R}^d$  and  $0 \leq s \leq 1$ . There has also been interest in nonlocal set interactions via cross interaction of two phases (cf. [6, 16]).

Finally, we remark that we only consider locally integrable kernels although kernels that are not locally integrable and appear in crystallization problems are of great interest from the point of view of the calculus of variations.

### 3. MASS SCALING

Throughout we consider nonlocal interaction energies (1.2) over three different classes:

- $\mathcal{S}_m :=$  Characteristic functions of measurable sets  $\Omega \subset \mathbb{R}^d$  with  $|\Omega| = m$ ;
- $\mathcal{A}_{m,M} := \left\{ \rho \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) : \|\rho\|_{L^1(\mathbb{R}^d)} = m \text{ and } 0 \leq \rho(x) \leq M \text{ a.e.} \right\}$ ;
- $\mathcal{P}(\mathbb{R}^d) :=$  probability measures over  $\mathbb{R}^d$ .

With an abuse of notation we denote the energy by  $E$  over each class; however, we emphasize the dependence on the admissible class using the notation  $E(\Omega)$ ,  $E(\rho)$  and  $E(\mu)$ , respectively, when needed. Note that minimization over  $\mathcal{S}_m$  is precisely our shape optimization problem **(P)**. Clearly  $\mathcal{S}_m \subset \mathcal{A}_{m,1}$  and  $\mathcal{A}_{m,1}$  is the weak closure of  $\mathcal{S}_m$  in the weak  $L^1$ -topology.

Over  $\mathcal{P}(\mathbb{R}^d)$  the minimal energy scales differently than on  $\mathcal{S}_m$  or  $\mathcal{A}_{m,M}$ . When we consider the nonlocal energy (1.2) over density functions  $\rho \in L^1(\mathbb{R}^d)$ , the shape of minimizers is independent of the mass  $m$ : The problem is homogeneous in  $\rho$ , that is

$$E(c\rho) = c^2 E(\rho)$$

for any  $c > 0$ .

On the other hand, for **(P)** this is not the case since the attractive and repulsive parts of the interaction energy scale differently under a dilation. To see this let us split the energy into its attractive and repulsive parts,  $E = E_q - E_p$ , where

$$E_q(\Omega) = \frac{1}{q} \int_{\Omega} \int_{\Omega} |x - y|^q dx dy \quad \text{and} \quad E_p(\Omega) = \frac{1}{p} \int_{\Omega} \int_{\Omega} |x - y|^p dx dy.$$

Given a measurable set  $\Omega \subset \mathbb{R}^d$  of volume  $m$ , and  $t > 0$ , the dilated set

$$t\Omega := \{x \in \mathbb{R}^d : t^{-1}x \in \Omega\}$$

has mass equal to  $t^d m$ . The attractive and repulsive parts of the energy satisfy

$$E_q(t\Omega) = t^{2d+q} E_q(\Omega) \quad \text{and} \quad E_p(t\Omega) = t^{2d+p} E_p(\Omega).$$

Choosing  $t = m^{-1/d}$  and replacing  $\Omega$  with  $t\Omega$ , we see that **(P)** is equivalent to minimizing

$$(3.1) \quad E(t\Omega) = m^{2+\frac{q}{d}} E_q(\Omega) - m^{2+\frac{p}{d}} E_p(\Omega) \quad \text{over sets of volume } |\Omega| = 1.$$

Since  $p < 0 < q$ , we see from (3.1) that for sets of large mass the energy is dominated by attraction, whereas for small mass it is dominated by repulsion. The separate effects of each term are characterized by the following well-known application of the Riesz rearrangement inequality.

**Proposition 3.1.** *For every non-zero  $r > -d$  and each  $m > 0$ , balls are the unique minimizers of the energy*

$$E_r(\Omega) = \frac{1}{r} \int_{\Omega} \int_{\Omega} |x - y|^r dx dy$$

*among measurable sets  $\Omega \subset \mathbb{R}^d$  of measure  $m$ . There is no maximum; the supremum takes the value  $+\infty$  for  $r > 0$ , and 0 for  $-d < r < 0$ .*

*Proof.* Given a set  $\Omega \subset \mathbb{R}^d$  of measure  $m > 0$ , let  $\Omega^*$  be the open ball of the same measure centered at the origin. Since the kernel  $K_r(x) = \frac{1}{r}|x|^r$  is radially increasing, it follows from the classical Riesz rearrangement inequality [7, 28] that

$$E_r(\Omega^*) \leq E_r(\Omega).$$

(Note that the sign of the factor  $\frac{1}{r}$  compensates for the change of monotonicity when  $r < 0$ .) Since  $K_r$  is strictly increasing, equality holds only if  $\Omega$  agrees with  $\Omega^*$  up to a translation and a set of measure zero [27], that is, if  $\Omega$  itself is a ball.

For the second statement, construct maximizing sequences of sets  $\{\Omega_n\}_{n \geq 1}$ , where each  $\Omega_n$  is union of  $n$  balls of mass  $m/n$  whose pairwise distance exceeds  $n$ .  $\square$

In light of (3.1), if the mass is large, the attractive interaction dominates and we expect that balls are global minimizers for **(P)**. If the mass is small, the repulsion dominates and we expect that minimizers fail to exist: Rather, a minimizing sequence converges weakly to a density function taking on values strictly between 0 and 1. We now make these statements precise.

#### 4. THE RELAXED PROBLEM

We consider the following relaxation of **(P)**:

$$\text{(RP)} \quad \text{Minimize} \quad E(\rho) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x-y) \rho(x) \rho(y) dx dy \quad \text{over} \quad \mathcal{A}_{m,1}.$$

In this section we will work with radially symmetric kernels  $K(\cdot)$  which are

$$(4.1) \quad \text{locally integrable, nonnegative, lower semicontinuous, and satisfy} \quad \lim_{|x| \rightarrow \infty} K(x) = \infty.$$

Note that this class of kernels include power-law potentials of the form (1.1).

The following existence result was first proved for power-law potentials in [13]. To obtain the existence of minimizers for more general kernels we can use the arguments in [33, Theorem 3.1] and obtain that a minimizing sequence is tight. Then combining this with the arguments in [13, Theorem 2.1] we can conclude that a minimizing sequence is compact, i.e., has a convergent subsequence in the class of admissible functions  $\mathcal{A}_{m,1}$ .

**Proposition 4.1 (Existence of solutions).** *Under the assumptions of (4.1), the problem (RP) admits a solution for each  $m > 0$ .*

We say that a function  $\rho$  is a *local minimizer* of  $E$  in  $\mathcal{A}_{m,1}$  (in the  $L^1$ -topology), if  $E(\rho) \leq E(\rho + \phi)$  for all  $\phi \in L^1(\mathbb{R}^d)$  with  $\|\phi\|_{L^1} < \delta$  and  $\rho + \phi \in \mathcal{A}_{m,1}$ . Local minimizers satisfy the following necessary condition.

**Lemma 4.2.** *Let  $\rho$  be a local minimizer of the energy  $E$  in  $\mathcal{A}_{m,1}$ . Then there exists a constant  $\lambda > 0$  such that (except for  $x$  in a set of measure zero),*

$$(4.2) \quad K * \rho(x) \quad \begin{cases} = \lambda & \text{if } 0 < \rho(x) < 1, \\ \geq \lambda & \text{if } \rho(x) = 0, \\ \leq \lambda & \text{if } \rho(x) = 1. \end{cases}$$

*Proof.* We proceed as in [21, Lemma 4.1.2] and [16, Lemma 1.2]. Let  $\rho \in \mathcal{A}_{m,1}$  be a local minimizer of  $\mathbf{E}$ . We need to construct perturbations that are nonnegative on  $S_0 := \{x : \rho(x) = 0\}$ , nonpositive on  $S_1 := \{x : \rho(x) = 1\}$ , and preserve mass. Let  $\phi$  and  $\psi \in L^1(\mathbb{R}^d)$  be compactly supported, bounded, nonnegative functions with  $\phi = 0$  a.e. in  $S_1$ ,  $\psi = 0$  a.e. in  $S_0$ , and

$$(4.3) \quad \int_{\mathbb{R}^d} \phi(x) dx = \int_{\mathbb{R}^d} \psi(x) dx = 1.$$

Fix  $\epsilon > 0$ , and define

$$\begin{aligned} \phi_\epsilon(x) &:= \frac{1}{\|\phi \chi_{\{1-\rho>\epsilon\}}\|_{L^1(\mathbb{R}^d)}} \phi(x) \chi_{\{1-\rho(x)>\epsilon\}}(x), \\ \psi_\epsilon(x) &:= \frac{1}{\|\psi \chi_{\{\rho>\epsilon\}}\|_{L^1(\mathbb{R}^d)}} \psi(x) \chi_{\{\rho(x)>\epsilon\}}(x). \end{aligned}$$

By construction,  $\rho + t(\phi_\epsilon - \psi_\epsilon)$  lies in  $\mathcal{A}_{m,1}$  and the perturbation is small for sufficiently small values of  $t > 0$ . Since  $\rho$  is a minimizer, it follows that

$$0 \leq \lim_{t \rightarrow 0^+} \frac{\mathbf{E}(\rho + t(\phi_\epsilon - \psi_\epsilon)) - \mathbf{E}(\rho)}{t} = 2 \int_{\mathbb{R}^d} K * \rho(x) (\phi_\epsilon - \psi_\epsilon)(x) dx.$$

Clearly,  $\phi_\epsilon \rightarrow \phi$  and  $\psi_\epsilon \rightarrow \psi$  as  $\epsilon \rightarrow 0$ . By dominated convergence, we can pass to the limit as  $\epsilon \rightarrow 0$  and obtain

$$(4.4) \quad \int_{\mathbb{R}^d} K * \rho(x) (\phi - \psi)(x) dx \geq 0.$$

By density, (4.4) holds for all nonnegative functions  $\phi, \psi$  in  $L^1(\mathbb{R}^d)$  with  $\phi(x) = 0$  on  $S_1$ ,  $\psi(x) = 0$  on  $S_0$ , and  $\|\phi\|_{L^1(\mathbb{R}^d)} = \|\psi\|_{L^1(\mathbb{R}^d)} = 1$ . Minimizing and maximizing separately over  $\phi$  and  $\psi$ , we obtain a constant  $\lambda \in \mathbb{R}$  such that

$$\inf \left\{ \int_{\mathbb{R}^d} K * \rho(x) \phi(x) dx : \|\phi\|_{L^1(\mathbb{R}^d)} = 1, \phi \geq 0, \text{ and } \phi = 0 \text{ a.e. on } S_1 \right\} \geq \lambda$$

and

$$\sup \left\{ \int_{\mathbb{R}^d} K * \rho(x) \psi(x) dx : \|\psi\|_{L^1(\mathbb{R}^d)} = 1, \psi \geq 0, \text{ and } \psi = 0 \text{ a.e. on } S_0 \right\} \leq \lambda.$$

In particular,  $\lambda > 0$  since so are  $K, \rho$  and  $\psi$ . We conclude that  $K * \rho \geq \lambda$  a.e. on  $\{x : \rho(x) < 1\}$ , and  $K * \rho \leq \lambda$  a.e. on  $\{x : \rho(x) > 0\}$ , as claimed.  $\square$

One consequence of Lemma 4.2 is that the minimizers of  $\mathbf{E}$  over  $\mathcal{A}_{m,1}$  are compactly supported. This fact was established in [10] for minimizers of  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$ ; a more direct approach was used in [16, Proposition 1.11]. In our situation, the argument is simple and we present it here for the convenience of the reader.

**Lemma 4.3.** *Under the assumptions of (4.1), every local minimizer for (RP) in  $\mathcal{A}_{m,1}$  has compact support.*

*Proof.* By Lemma 4.2, there exists a constant  $\lambda$  such that  $K * \rho \leq \lambda$  almost everywhere on the support of  $\rho$ . Changing  $\rho$  on a set of measure zero, if necessary, we may assume that  $K * \rho(x) \leq \lambda$  for all  $x$  with  $\rho(x) > 0$ .

Let  $R > 0$  be large enough such that

$$C_R := \int_{|y|<R} \rho(y) dy > 0.$$

Since  $K$  and  $\rho$  are nonnegative, we have for  $x \in \mathbb{R}^d$  that

$$\begin{aligned} K * \rho(x) &\geq \int_{|y| < R} K(x-y) \rho(y) dy \\ &\geq C_R \inf\{K(z) : |z| > |x| - R\}. \end{aligned}$$

Therefore

$$\lim_{|x| \rightarrow \infty} K * \rho(x) = \infty,$$

and the sub-level set  $\{x : K * \rho \leq \lambda\}$  is bounded. Since the sub-level set contains the support of  $\rho$ , the claim follows.  $\square$

A useful consequence of Lemma 4.3 is that  $K * \rho$  is continuous (since  $K$  is locally integrable). We can now reduce the geometric variational problem to the relaxed problem.

**Theorem 4.4 (Necessary and sufficient conditions for existence of  $(\mathbf{P})$ ).** *Let  $K$  be a radially symmetric kernel satisfying (4.1). Then the problem  $(\mathbf{P})$  has a solution  $\Omega \subset \mathbb{R}^d$  if and only if its characteristic function  $\chi_\Omega$  is a solution of  $(\mathbf{RP})$ .*

*Proof.* We will show that

$$(4.5) \quad \inf_{|\Omega|=m} \mathbf{E}(\Omega) = \inf_{\rho \in \mathcal{A}_{m,1}} \mathbf{E}(\rho)$$

and establish a relationship between the solutions of the two variational problems. The inequality  $\geq$  is trivial from the definition of the two variational problems: the characteristic function  $\chi_\Omega$  of any set  $\Omega \subset \mathbb{R}^d$  of measure  $m$  lies in  $\mathcal{A}_{m,1}$ . Similarly if  $\chi_\Omega$  is a global minimizer for  $\mathbf{E}$ , then clearly  $\Omega$  is global minimizer for  $(\mathbf{P})$ .

Conversely, suppose that the global minimum of  $\mathbf{E}$  over  $\mathcal{A}_{m,1}$  is not achieved by a characteristic function, and fix a global minimizer  $\rho$ . By Lemma 4.3,  $\rho$  has compact support. Choose a sequence of measurable sets  $\{\Omega_n\}_{n \geq 1}$  whose characteristic functions  $\rho_n = \chi_{\Omega_n}$  converge to  $\rho$  weakly in  $L^1(\mathbb{R}^d)$ . To be specific, take a dyadic decomposition of  $\mathbb{R}^d$  into cubes of side length  $2^{-n}$ , and let the intersection of  $\Omega_n$  with a given cube  $Q$  be the centered closed subcube of volume  $\int_Q \rho(x) dx$ . By construction,  $|\Omega_n| = m$ , and  $\rho_n \in \mathcal{A}_{m,1}$ . Since  $\rho$  has compact support, the sets  $\Omega_n$  are contained in a common compact set.

Clearly,  $\rho_n \rightharpoonup \rho$  weakly in  $L^1(\mathbb{R}^d)$ . It follows from the local integrability of  $K$  that

$$\lim_{n \rightarrow \infty} K * \rho_n(x) = K * \rho(x)$$

for every  $x \in \mathbb{R}^d$ , that is,  $K * \rho_n$  converges pointwise to  $K * \rho$ . By dominated convergence,  $K * \rho_n \rightarrow K * \rho$  strongly in  $L^1(\mathbb{R}^d)$ . Using once more that  $\rho_n \rightharpoonup \rho$ , we conclude that

$$\mathbf{E}(\Omega_n) = \int_{\Omega_n} K * \rho_n dx \rightarrow \int_{\mathbb{R}^d} (K * \rho) \rho dx = \mathbf{E}(\rho).$$

In particular,

$$\inf_{|\Omega|=m} \mathbf{E}(\Omega) \leq \mathbf{E}(\rho) = \min_{\rho \in \mathcal{A}_{m,1}} \mathbf{E}(\rho),$$

and  $\{\Omega_n\}$  is a minimizing sequence for  $(\mathbf{P})$ . Since  $\mathbf{E}(\Omega) > \mathbf{E}(\rho)$  for every  $\Omega \subset \mathbb{R}^d$ , no minimizer exists.  $\square$



5. THE CASE OF  $q = 2$ 

In this section we specialize to kernels of the form (1.1) where the attractive term is quadratic, i.e.,  $q = 2$ . The key observation here is that **(RP)** can be rewritten as a convex minimization problem in the parameter regime  $q = 2$  and  $-d < p < 0$ , hence, allowing us to conclude the uniqueness of minimizers of the relaxed problem.

**Lemma 5.1.** *For  $q = 2$  and  $-d < p < 0$ , the solution of problem **(RP)** is unique up to translation, and is given by a radial function.*

*Proof.* Since the energy  $E(\rho)$  is translation invariant, without loss of generality, we assume that  $\int_{\mathbb{R}^d} x\rho(x) dx = 0$ . Then

$$E_q(\rho) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^2 \rho(x)\rho(y) dx dy = m \int_{\mathbb{R}^d} |x|^2 \rho(x) dx,$$

and the attractive part of the energy is linear in  $\rho$ .

On the other hand, when  $-d < p < 0$ , the repulsive part of the energy

$$-E_p(\rho) = -\frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^p \rho(x)\rho(y) dx dy$$

is strictly convex over  $\mathcal{A}_{m,1}$  since the Fourier transform of the kernel  $-K_p(x) = -\frac{1}{p}|x|^p$  is strictly positive when  $-d < p < 0$  [28, Corollary 5.10].

Therefore the energy is strictly convex among all functions in  $\mathcal{A}_{m,1}$  with zero first moments, and the solution of **(RP)** is unique up to translations.

Radial symmetry of the solution follows from the uniqueness and, due to its isotropic nature, the rotational symmetry of the energy  $E(\rho)$  around the center of mass of any  $\rho \in \mathcal{A}_{m,1}$ .  $\square$

*Remark 5.2.* For  $x \in \mathbb{R}^2$  we take

$$K(x) = \frac{1}{2}|x|^2 - \log|x|$$

when  $p = 2 - d$ , and the repulsive part of the energy is given by

$$-E_p(\rho) = - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|) \rho(x)\rho(y) dx dy = C \|\rho\|_{H^{-1}}^2.$$

Hence, the repulsion term is strictly convex and we still have the uniqueness of minimizers in the case  $p = 2 - d$  when  $d = 2$ .

**5.1. Nonexistence for **(P)** for small mass.** To prove the nonexistence of minimizers in the small mass regime we specialize to kernels of the form (1.1) with  $q = 2$  and  $-d < p \leq 2 - d$ . This range of Riesz potentials share some important properties via their correspondence to the obstacle problem for  $(-\Delta)^s$  with  $s \in (0, 1]$  which enjoys rather strong regularity features [9, 32]. This connection between the obstacle problem and nonlocal interaction energies over  $\mathcal{P}(\mathbb{R}^d)$  was recently exploited by Carrillo, Delgadino and Mellet [12] to obtain regularity of local minimizers with respect to the  $\infty$ -Wasserstein metric  $d_\infty$ .<sup>2</sup> Although, a priori local minimizers in the  $d_\infty$ -topology are not comparable with the local minimizers in the  $L^1$ -topology the regularity result

<sup>2</sup>For  $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$  the  $\infty$ -Wasserstein metric is defined as

$$d_\infty(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \sup_{(x, y) \in \text{supp } \pi} |x - y|,$$

where  $\Pi(\mu, \nu) := \{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \pi(A \times \mathbb{R}^d) = \mu(A) \text{ and } \pi(\mathbb{R}^d \times A) = \nu(A) \text{ for all } A \subset \mathbb{R}^d\}$ .

is true for *global* minimizers independent of the topology. Here we rephrase their results for interaction potentials in power-law form (1.1) (cf. [12, Remark 3.1]).

**Lemma 5.3** (Theorems 3.4 and 3.10 in [12]). *Let  $K$  be given by (1.1). Let  $\mu \in \mathcal{P}(\mathbb{R}^d)$  be a local minimizer of  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$  in the topology induced by  $d_\infty$ .*

- (i) *If  $q > 0$  and  $p = 2 - d$ , then  $\mu$  is absolutely continuous with respect to the Lebesgue measure and there exists a function  $\phi \in L^\infty(\mathbb{R}^d)$  such that  $d\mu(x) = \phi(x) dx$ .*
- (ii) *If  $q > 0$  and  $p < 2 - d$ , then  $\mu$  is absolutely continuous with respect to the Lebesgue measure and there exists a function  $\phi \in C^\alpha(\mathbb{R}^d)$  for all  $\alpha < 1$  such that  $d\mu(x) = \phi(x) dx$ .*

*Remark 5.4* ( $L^\infty$ -control on global minimizers). In the parameter regime  $q > 0$  and  $-d < p \leq 2 - d$  we can still control the  $L^\infty$ -bound of a *global* minimizer. In fact, [10, Theorem 1.4] implies that any global minimizer  $\mu \in \mathcal{P}(\mathbb{R}^d)$  of  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$  is compactly supported. This, in light of Lemma 5.3(ii), yields that the density function  $\phi$  is in  $L^\infty(\mathbb{R}^d)$ .

Using these results we can relate the  $L^\infty$ -bound of minimizers to the mass constraint  $m$  via scaling which in turn enables us to obtain nonexistence of minimizers of the set energy  $\mathbf{E}(\Omega)$  when the mass is sufficiently small.

**Proof of Theorem 1.1(i):** Let  $\mu \in \mathcal{P}(\mathbb{R}^d)$  be a global minimizer of  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$ . Such a minimizer exists by [10, Theorem 1.4] or [33, Theorem 3.1] in the parameter regime  $q = 2$ ,  $-d < p \leq 2 - d$ . By Lemma 5.3 and Remark 5.4,  $\mu$  is absolutely continuous with respect to the Lebesgue measure with bounded density, i.e., there exists a constant  $C > 0$  such that  $\|\mu\|_{L^\infty} < C$  with an abuse of notation.

Consider  $\rho_m := m\mu$ . For  $m > 0$  sufficiently small we have that  $\rho_m \in \mathcal{A}_{m,1}$ . Now we claim that  $\rho_m$  minimizes  $\mathbf{E}$  over  $\mathcal{A}_{m,1}$ . To see this let  $\phi \in \mathcal{A}_{m,1}$  be an arbitrary function and note that  $(1/m)\phi \in \mathcal{P}(\mathbb{R}^d)$ . Using the fact that  $\mu$  minimizes  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$  and the scaling of the energy  $\mathbf{E}$  we have that

$$\mathbf{E}(\rho_m) = m^2 \mathbf{E}(\mu) \leq m^2 \mathbf{E}\left(\frac{1}{m}\phi\right) = \mathbf{E}(\phi).$$

On the other hand, by Lemma 5.1 and Remark 5.2,  $\rho_m$  is the unique minimizer of  $\mathbf{E}$  over  $\mathcal{A}_{m,1}$  in any dimension  $d \geq 2$ . For  $m$  sufficiently small we have  $\|\rho_m\|_{L^\infty(\mathbb{R}^d)} = m\|\mu\|_{L^\infty(\mathbb{R}^d)} \leq mC < 1$ . Hence, when  $m$  is small  $\rho_m$  is not a characteristic function of a set. Since it is the unique solution to the problem **(RP)** by Theorem 4.4 the energy  $\mathbf{E}$  does not admit a minimizer over measurable sets of measure  $m$ .  $\square$

**5.2. Existence for **(P)** for large mass.** We first note that *heuristically* Lemma 4.2 and Theorem 4.4 should imply existence for  $m \geq \omega_d$  in the case of Newtonian repulsion  $p = 2 - d$  and quadratic attraction  $q = 2$ . To see this formally, assume that any local minimizer of **(RP)** is continuous on its support and let

$$\Omega = \{x \in \mathbb{R}^d : 0 < \rho(x) < 1\}$$

for a local minimizer  $\rho$ . Suppose, for a contradiction, that  $|\Omega| > 0$ . Since we assume that  $\rho$  is continuous on its support,  $\Omega$  is an open set. Lemma 4.2 implies there exists a constant  $\lambda$  such that

$$K * \rho(x) = \lambda \quad \text{on } \Omega.$$

Taking the Laplacian of both sides, we find for all  $x \in \Omega$ ,

$$\begin{aligned} \Delta K * \rho(x) &= \frac{1}{2} \Delta (|\cdot|^2 * \rho)(x) + \frac{1}{d-2} \Delta \left( \frac{1}{|\cdot|^{d-2}} * \rho \right)(x) \\ &= d \int_{\mathbb{R}^d} \rho(y) dy - d\omega_d \rho(x) = 0, \end{aligned}$$

or

$$\frac{m}{w_d} = \rho(x).$$

Hence if  $m \geq \omega_d$  we obtain a contradiction unless the set  $\Omega$  is empty. This shows that for  $m \geq \omega_d$ , every local minimizer of **(RP)** must be a characteristic function. By Theorem 4.4, this establishes existence of **(P)** for  $m \geq \omega_d$  and characterizes the minimizer. We will shortly prove this result rigorously and show that this lower bound is sharp.

We now turn to the full range of Riesz potentials, i.e., to the regime  $-d < p < 0$ . To prove the existence of set minimizers for the energy  $E$  when the mass  $m$  is sufficiently large we will first prove that the characteristic function of a ball is indeed a critical point of the relaxed problem **(RP)**.

**Lemma 5.5** (Large balls satisfy the necessary condition of Lemma 4.2). *Let any  $q > 1$  and  $-d < p < 0$ . For sufficiently large mass  $m$ , the characteristic function of a ball of mass  $m$  is a critical point for the energy  $E$  on  $\mathcal{A}_{m,1}$ .*

*Proof.* We split the kernel into its attractive and repulsive parts by defining  $K_q := (1/q)|x|^q$  and  $K_p := (1/|p|)|x|^p$  so that  $K = K_q + K_p$ . Let  $R$  be the radius of the ball of mass  $m$ . Since  $K_q$  and  $K_p$  are radial, so are  $K_q * \chi_{B_R}$  and  $K_p * \chi_{B_R}$ .

Since  $K_q$  is radially increasing, so is  $K_q * \chi_{B_R}$ . For  $|x| \geq R/2$ , we can estimate the radial derivative by

$$(5.1) \quad \begin{aligned} \left( \nabla (K_q * \chi_{B_R})(x) \cdot \frac{x}{|x|} \right) &= \int_{|y| \leq R} |x-y|^{q-2} (x-y) \cdot \frac{x}{|x|} dy \\ &\geq C_q R^{d+q-1}, \end{aligned}$$

where the constant

$$C_q = \inf_{t \geq \frac{1}{2}} \int_{|y| \leq 1} |te_1 - y|^{q-2} (t - y_1) dy$$

is positive since  $q > 1$  and  $e_1$  denotes a unit vector in  $\mathbb{R}^d$ .

Similarly,  $K_p * \chi_{B_R}$  is a decreasing function of  $|x|$ , and we estimate for  $|x| \geq R/2$ ,

$$(5.2) \quad \left( \nabla (K_p * \chi_{B_R})(x) \cdot \frac{x}{|x|} \right) \geq -C_p R^{d+p-1}$$

for some constant  $C_p > 0$ .

Let  $R$  be sufficiently large so that  $C_q R^q > C_p R^p$ . Such a number  $R$  exists since  $p < q$ . From (5.1) and (5.2) we get that  $(K_q + K_p) * \chi_{B_R}(x)$  is increasing in  $|x|$  for  $|x| \geq R/2$ . Therefore

$$K * \chi_{B_R}(x) \geq \lambda_R := K * \chi_{B_R}(x) \Big|_{|x|=R}$$

for  $|x| \geq R$ . Furthermore,

$$(5.3) \quad K * \chi_{B_R}(x) < \lambda_R$$

for  $R/2 \leq |x| < R$ .

We need to show that (5.3) extends to  $|x| < R/2$ . We first note that since both  $K_q * \chi_{B_R}$  and  $K_p * \chi_{B_R}$  are radially symmetric we have that

$$\begin{aligned}
 \lambda_R &= \int_{|y| \leq R} \frac{|Re_1 - y|^q}{q} + \frac{|Re_1 - y|^p}{|p|} dy \\
 (5.4) \quad &= R^{d+q} \int_{|y| \leq 1} \frac{|e_1 - y|^q}{q} dy + R^{d+p} \int_{|y| \leq 1} \frac{|e_1 - y|^p}{|p|} dy \\
 &= \tilde{C}_q R^{d+q} + \tilde{C}_p R^{d+p}
 \end{aligned}$$

where  $\tilde{C}_q = K_q * \chi_{B_1}(x) \Big|_{|x|=1} > 0$  and  $\tilde{C}_p = K_p * \chi_{B_1}(x) \Big|_{|x|=1} > 0$ .

Using the fact that  $K_q * \chi_{B_R}$  is increasing in  $|x|$  and  $K_p * \chi_{B_R}$  is decreasing in  $|x|$ , we estimate

$$\begin{aligned}
 (K * \chi_{B_R})(x) &\leq (K_q * \chi_{B_R})(x) \Big|_{|x|=R/2} + (K_p * \chi_{B_R})(0) \\
 &= \tilde{\tilde{C}}_q R^{d+q} + \tilde{\tilde{C}}_p R^{d+p},
 \end{aligned}$$

where

$$\tilde{\tilde{C}}_q := K_q * \chi_{B_1}(x) \Big|_{|x|=1/2}.$$

Hence,  $\tilde{\tilde{C}}_q < \tilde{C}_q$  as  $K_q * \chi_{B_R}$  is radially increasing. Comparing this inequality with (5.4), we see that (5.3) also holds for  $|x| \leq R/2$ , if  $R$  is sufficiently large.  $\square$

**Proof of Theorem 1.1(ii):** By Lemmas 5.1 and 5.5 the function  $\chi_{B(0,R)}$  with  $R = (m/\omega_d)^{1/d}$  is up to translation the unique solution of **(RP)** for  $q = 2$  and  $-d < p < 0$  provided  $m$  is sufficiently large. By convexity, it must be a global minimizer.  $\square$

Finally, as we noted in the introduction, in the case of Coulomb repulsion, i.e., when  $p = 2 - d$ , the thresholds of mass for existence/nonexistence appearing in Theorems 1.1 (i) and (ii) coincide and can be computed explicitly. This provides the complete picture regarding the minimization of  $\mathbf{E}$  either over  $\mathcal{S}_m$  or  $\mathcal{A}_{m,1}$  in this special regime.

**Proof of Theorem 1.1 (iii).** Consider the relaxed energy  $\mathbf{E}$  over  $\mathcal{A}_{m,1}$ , and let  $\rho_R := \chi_{B(0,R)}$  with  $R = (m/\omega_d)^{1/d}$  and  $\rho_1 := (m/\omega_d)\chi_{B(0,1)}$ . Note that both  $\rho_R$  and  $\rho_1$  are in  $\mathcal{A}_{m,1}$ .

Using the fact that  $(d-2)^{-1} \int_{B(0,R)} |x-y|^{2-d} dy = d\omega_d \Phi(x)$  where  $\Phi(x)$  solves the equation  $-\Delta \Phi = \rho_R$  on  $\mathbb{R}^d$  we can explicitly compute that

$$K * \rho_R(x) = \begin{cases} \frac{m-\omega_d}{2} |x|^2 + \frac{d\omega_d R^2}{2(d-2)} + \frac{dmR^2}{2(d+2)} & \text{if } |x| \leq R, \\ \frac{m}{2} |x|^2 + \frac{\omega_d R^d}{(d-2)} |x|^{2-d} + \frac{dmR^2}{2(d+2)} & \text{if } |x| > R. \end{cases}$$

This shows via (4.2) that  $\rho_R$  is a critical point of  $\mathbf{E}$  over  $\mathcal{A}_{m,1}$  if and only if  $m \geq \omega_d$ . Then by Lemma 5.1 we get that  $\rho_R$  is the unique minimizer of  $\mathbf{E}(\rho)$  if and only if  $m \geq \omega_d$ . On the other hand, when  $m < \omega_d$  a simple calculation shows that  $\mathbf{E}(\rho_1) < \mathbf{E}(\rho_R)$ . Moreover, by [13, Theorem

2.4],  $\rho_1$  is the unique global minimizer of  $\mathbf{E}$  over  $\mathcal{A}_{m,1}$  when  $m < \omega_d$ . Hence, the result follows by Theorem 4.4.  $\square$

*Remark 5.6* (Failure of minimality of balls in 2-dimensions). For more singular repulsive powers in 2-dimensions, we can determine the threshold below which the ball fails to be the global minimizer of  $\mathbf{E}(\rho)$  by explicit calculations. When  $d = 2$ ,  $q = 2$  and  $-2 < p < 0$ , the energy of a ball of radius  $R = (m/\pi)^{1/2}$  is given by

$$\mathbf{E}(\chi_{B(0,R)}) = \frac{\pi}{2}R^6 + \frac{2\pi^2\Gamma(2+p)}{(-p)\Gamma(2+\frac{p}{2})\Gamma(3+\frac{p}{2})}R^{4+p},$$

where  $\Gamma$  denotes the  $\Gamma$ -function. The computation of the attractive part of the energy is trivial; the computation of the repulsive part is given in [25, Corollary 3.5]. On the other hand,

$$\mathbf{E}(R^2\chi_{B(0,1)}) = \left( \frac{\pi}{2} + \frac{2\pi^2\Gamma(2+p)}{(-p)\Gamma(2+\frac{p}{2})\Gamma(3+\frac{p}{2})} \right) R^4.$$

Thus, choosing  $R_c$  so that

$$\frac{\pi}{2}R_c^2 + \left( \frac{2\pi^2\Gamma(2+p)}{(-p)\Gamma(2+\frac{p}{2})\Gamma(3+\frac{p}{2})} \right) R_c^p > \frac{\pi}{2} + \frac{2\pi^2\Gamma(2+p)}{(-p)\Gamma(2+\frac{p}{2})\Gamma(3+\frac{p}{2})},$$

and noting that  $R_c < 1$  we see that for any  $R \leq R_c$  we have that

$$\mathbf{E}(R^2\chi_{B(0,1)}) \leq \mathbf{E}(\chi_{B(0,R)});$$

hence,  $\chi_{B(0,R)}$  is not a global minimizer of  $\mathbf{E}$  over  $\mathcal{A}_{m,1}$ .

## 6. THE REGIME OF $q > 0$

As we noted before the quadratic attraction case is special as the attractive part of the energy either over  $\mathcal{A}_{m,1}$  or  $\mathcal{P}(\mathbb{R}^d)$  is linear in its argument when we fix the center of mass of competitors to zero. This allows us to conclude the uniqueness of solutions to **(RP)**. The uniqueness of minimizers is key to the existence of solutions to **(P)** as we utilize this to conclude that any stationary state to **(RP)** has to minimize the energy  $\mathbf{E}$  over  $\mathcal{A}_{m,1}$ . When  $q \neq 2$ , on the other hand, even though Lemma 5.5 shows that the balls are stationary states in the parameter regime  $q > 1$ ,  $-d < p < 0$  when  $m > 0$  is large, due to the possible lack of uniqueness of minimizers, we cannot conclude the existence of solutions to **(P)** for large measure. Nevertheless, we believe that the problem **(P)** admits a solution for large values  $m > 0$  when  $q > 1$  as the energy is dominated by the attractive term which is minimized by balls of measure  $m$ .

The uniqueness of minimizers is also an important ingredient in establishing nonexistence of solutions to **(P)**. Indeed, it is the uniqueness of solutions to **(RP)** which allows us to conclude that any solution of **(RP)** can be written as  $m\mu$  for some  $\mu$  that minimizes  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$ . Intuitively, for small  $m > 0$ , the  $L^\infty$ -bound in the problem **(RP)** is not active, and the morphology of minimizers should be the same as of those over  $\mathcal{P}(\mathbb{R}^d)$ . When  $m > 0$  is large, on the other hand, the  $L^\infty$ -bound becomes active and adds additional repulsive effects to the problem penalizing accumulations.

When  $q > 0$  and  $-d < p \leq 2 - d$ , nonexistence of solutions to **(P)** as in Theorem 1.1(ii) would also be true if the  $L^\infty$ -bound found in Lemma 5.3 and Remark 5.4 was uniform for *any* measure minimizer  $\mu$ . In that case, the proof of Theorem 1.1(ii) would translate almost verbatim to the power regime  $q > 0$ ,  $-d < p \leq 2 - d$ . A result in this direction is the following.

**Proposition 6.1.** *Let  $K$  be of the form (1.1). Then for  $q > 0$ ,  $-d < p < 0$ , and for  $m > 0$  sufficiently small the ball of measure  $m$  is not a solution of  $(\mathbf{P})$ .*

*Proof.* We will proceed by contradiction. If  $B(0, r_n)$  with  $\omega_d r_n^d = 1/n$  were solutions of  $(\mathbf{P})$  with  $m = 1/n$  for any  $n \in \mathbb{N}$  then the weak limit of the sequence  $\rho_n = n\chi_{B(0, r_n)} \in \mathcal{P}(\mathbb{R}^d)$  would also minimize the energy  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$ . This follows by noting that for fixed  $\mu$  that globally minimizes  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$  we have that for sufficiently large  $n \in \mathbb{N}$

$$\mathbf{E}(\mu) \leq \mathbf{E}(\rho_n) = n^2 \mathbf{E}(\chi_{B(0, r_n)}) \leq n^2 \mathbf{E}(n^{-1}\mu) = \mathbf{E}(\mu).$$

The second inequality follows from (4.5). Thus  $\lim_{n \rightarrow \infty} \mathbf{E}(\rho_n) = \inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} \mathbf{E}(\mu)$ , i.e.,  $\{\rho_n\}_{n \in \mathbb{N}}$  is a minimizing sequence for the energy  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$ . Arguing as in [33, Theorem 3.1] via Lions' Concentration Compactness Theorem we obtain that  $\rho_n$  has a weakly convergent subsequence and by the weak lower semicontinuity of  $\mathbf{E}$  its limit minimizes  $\mathbf{E}$  over  $\mathcal{P}(\mathbb{R}^d)$ . However, as  $n \rightarrow \infty$ ,  $\{\rho_n\}_{n \in \mathbb{N}}$  converges weakly to  $\delta_0$ , the Dirac measure at  $x = 0$ , which has infinite energy.  $\square$

A possible way of generalizing this result to conclude nonexistence of  $(\mathbf{P})$  for small  $m$  is via the *energy-per-particle-pair*

$$(6.1) \quad \eta(m) := \inf_{\rho \in \mathcal{A}_{m,1}} \frac{\mathbf{E}(\rho)}{m^2}$$

associated with  $(\mathbf{RP})$ . Because of the positivity of  $K$ , it is easy to see that if  $(\mathbf{P})$  admits a solution for all  $m > 0$ , then  $\eta(m)$  is nondecreasing in  $m$ . Moreover, if  $\eta(m)$  is strictly increasing in  $m$  (which is true when  $q = 2$ ,  $-d < p < 0$ ) then we would have the following sufficient condition for nonexistence of minimizers: If  $\eta'(m_c) = 0$  for some  $m_c > 0$ , then  $(\mathbf{P})$  does not have a solution for  $m < m_c$ . Together with Lemma 5.3 and Remark 5.4, this would prove nonexistence of  $(\mathbf{P})$  for sufficiently small  $m > 0$  when  $q > 0$  and  $-d < p \leq 2 - d$ . These remarks highlight the fact that the (strict) monotonicity of  $\eta$  determines whether the  $L^\infty$ -constraint in  $\mathcal{A}_{m,1}$  is active for the given value of  $m$ .

Finally, it remains to be proved whether there exists a regime of  $m$ ,  $q$  and  $p$  where the minimizers are not balls. When  $q$  is sufficiently large we expect that solutions to  $(\mathbf{P})$  are rings rather than balls. Formally, the sequence of energies  $\{\mathbf{E}(\rho)\}_{q>0}$  converges to

$$\mathbf{E}_\infty(\rho) = \begin{cases} -\frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^p \rho(x) \rho(y) dx dy & \text{if } \text{diam}(\text{supp } \rho) \leq 1, \\ +\infty & \text{otherwise} \end{cases}$$

as  $q \rightarrow \infty$ . Due to the purely repulsive effects in the energy  $\mathbf{E}_\infty$  its minimizers  $\rho$  should have convex supports and accumulate on the boundary of  $\text{supp } \rho$ ; however, these questions are open even in the Newtonian case.

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