THE LIQUID DROP MODEL WITH A YUKAWA POTENTIAL: EXISTENCE OF MINIMIZERS AND SHARP STABILITY OF THE BALL

LIA BRONSARD, KENNETH DEMASON, AND IHSAN TOPALOGLU

ABSTRACT. We study a long-range perturbation of the perimeter functional by a nonlocal repulsive term defined through the Yukawa kernel, minimized under a volume constraint. Because the kernel decays exponentially, the competition between surface tension and repulsion is governed by two independent quantities, the screening rate and the volume. Specifically we prove that (i) above an explicit critical screening rate, minimizers exist at every volume, whereas earlier work required the volume to be large; (ii) below an explicit volume threshold minimizers exist for every screening rate; (iii) at small volume, and uniformly in the screening rate, the ball is the unique minimizer up to translation; and (iv) there exists a sharp volume threshold, depending on the screening rate, for the ball to be a stable volume-constrained critical point. The threshold is given in closed form for every screening rate and reduces to the known unscreened value when $\alpha = 0$. Our proofs overcome new technical changes due to the lack of homogeneity in the Yukawa kernel.

1. INTRODUCTION

In this paper we are interested in the stability and global/local minimality of balls B_r for the energy functional

$$\mathcal{F}_\alpha(E) \coloneqq \mathcal{P}(E) + \int_E \int_E \frac{e^{-\alpha|x-y|}}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$

in the class of sets of finite perimeter $E \subset \mathbb{R}^3$ such that $|E| = m$. Here $\alpha, m > 0$ and $\mathcal{P}(E)$ denotes the perimeter of the set E in the De Giorgi sense, see for instance [Mag12, Chapter 12]. The nonlocal term is given by

$$\mathcal{N}_\alpha(E) \coloneqq \int_E \int_E Y_\alpha(x-y)\,\mathrm{d}x\,\mathrm{d}y, \qquad Y_\alpha(z) \coloneqq \frac{e^{-\alpha|z|}}{|z|} \tag{1.1}$$

where Y_α is the Yukawa kernel.

The kernel Y_α arises in models where a charged phase carrying an interfacial tension is immersed in a medium of mobile opposite charges. The mobile charges rearrange to cancel the charge they surround, and to first order this adds a restoring term to the electrostatic equation: the potential solves $(-\Delta + \alpha^2)v = 4\pi\rho$ rather than $-\Delta v = 4\pi\rho$. Its Green function is $(4\pi)^{-1}Y_\alpha$, so pairwise interactions decay exponentially beyond the length $\ell = 1/\alpha$ while remaining Coulombic below it. For example, in the inner crust of a neutron star the medium is a gas of degenerate electrons, in which protons form clusters surrounded by a dilute neutron

Date: August 19, 2026.
2020 *Mathematics Subject Classification.* 49Q10, 49Q20, 49K21, 70G75, 82B21, 82B24.
Key words and phrases. Stability, Second variation, Yukawa kernel, Screened Coulomb, Global and local minimizers.

gas. The Thomas–Fermi linearization of the electron response makes ℓ the electron Thomas–Fermi length, and two protons at distance r then interact through $q^2 e^{-r/\ell}/r = q^2 Y_\alpha(r)$, with q being the proton charge (see for example [CH17, CH10]). Minimizing the associated free energy predicts the cluster shape, which as the density increases passes from near-spherical nuclei through rods, slabs, tubes and bubbles. This sequence of phases is referred to as nuclear pasta (see [RPW83, HSY84, PR95]). A second setting in which functionals of such form appear is regarding models for a charged phase in an electrolyte. There the medium consists of dissolved ions and the linearization is the Debye–Hückel approximation. In these models ℓ is the Debye length, set by the ionic strength, and it is adjustable by adding salt into the mixture. The screened repulsion obtained in this way is one of the two interactions in the classical theory of colloidal stability of Derjaguin, Landau, Verwey and Overbeek, in which it competes with a van der Waals attraction to determine whether suspended particles aggregate (see [Isr11, VO48]).

When $\alpha = 0$ the minimization problem with $Y_0(z) = |z|^{-1}$ among $E \subset \mathbb{R}^3$ with fixed mass is Gamow's liquid drop model of the atomic nucleus, and it has been long conjectured that minimizers are balls below a critical mass and that no minimizers exist above this threshold (see [CMT17] for a review on the topic). This conjecture has been resolved only very recently (see [CG26]). Screening the interaction changes this picture qualitatively, and the resulting minimization problem with Y_α is the object of very recent activity (cf. [BMP25, MNS25]).

The first author, Merlet, and Pegon [BMP25] have shown that the conjecture *fails* when the interaction is screened: for suitable values of the screening parameter they show that there exists a connected minimizer in $\mathbb{R}^3$ which is not a ball, and balls fail to minimize at that mass. Their setting is exactly the one studied here, their Yukawa potential $Y_{1,\kappa}(x) = e^{-|x|/\kappa}/|x|$ corresponds to our Y_α with $\kappa = 1/\alpha$, so that their functional is $\mathcal{F}_{1/\kappa}$. In two dimensions, Muratov, Novaga and Simon [MNS25] analyze the same Yukawa profile in the large-mass limit for a certain regime of screening parameters, where the nonlocal term concentrates on the boundary and cancels the perimeter to leading order. They show that the next-order Γ-limit is a weighted sum of the perimeter and Euler's elastica functional evaluated on the system of boundary curves, and the minimizers are disks above a critical screening rate and rings below it. For a general class of integrable kernels, Pegon [Peg21] proves the existence of minimizers above an explicit critical screening rate, provided the mass is large enough.

In this paper we first establish the existence of minimizers of $\mathcal{F}_\alpha$ for any volume m when the screening parameter is above a critical threshold. We follow the strategy in [MNS25] and extend the existence result in [Peg21] to the full range of m.

Theorem 1.1. *For $\alpha > (2\pi)^{1/3}$ and for any $m > 0$, the problem*

$$e_\alpha(m) \coloneqq \inf \Big\{ \mathcal{F}_\alpha(E) \, : \, |E| = m \Big\} \tag{1.2}$$

admits a solution. Moreover $m \mapsto m^{-1} e_\alpha(m)$ is strictly decreasing.

Our second theorem shows the existence of minimizers for any α provided m is sufficiently small. Using the arguments of Frank and Nam [FN21] for the liquid drop model ($\alpha = 0$ case), we prove the following.

Theorem 1.2. *Let*

$$\overline{m} := \frac{5(2^{1/3}-1)}{2(1-2^{-2/3})}.$$

Then for any $\alpha \geqslant 0$ and $m \in (0, \overline{m})$, the problem (1.2) admits a solution.

Next we turn to characterizing minimizers. Arguing as in Carazzato, Fusco, and Pratelli [CFP23], we prove that for sufficiently small volume, the global minimizer is the unit ball.

Theorem 1.3. *There exists $m_0 > 0$ (independent of α) such that for any $\alpha \geqslant 0$ and $m < m_0$, the unique minimizer, up to translations, of $\mathcal{F}_\alpha$ over sets $|E| = m$ is the ball $B[m]$ of volume m.*

Finally, we derive a critical stability threshold for balls of volume m.

Theorem 1.4. *For any $\alpha \geqslant 0$ there exists a computable $m_*(\alpha) > 0$ such that if $m > 0$ the ball $B[m]$ of volume m is a volume-constrained stable critical point of $\mathcal{F}_\alpha$ if and only if $m \leqslant m_*(\alpha)$. In particular, when $\alpha \geqslant (2\pi)^{1/3}$ we have $m_*(\alpha) = +\infty$, so that $B[m]$ is always stable; when $\alpha < (2\pi)^{1/3}$ we have $m_*(\alpha) \geqslant 5$.*

As a consequence of this stability result, using the well-established scheme in the literature (see for instance [AFM13, BC14, FFM+15]), we obtain the L^1-local minimality of $B[m]$. We note that our proof does not require any homogeneity, in contrast to [BC14, FFM+15].

Theorem 1.5. *For every $\alpha \geqslant 0$ and $m < m_*(\alpha)$ there exists $\varepsilon_*(\alpha, m) > 0$ such that if $|E| = m$ and $|E \Delta B[m]| \leqslant \varepsilon_* m$ then*

$$\mathcal{F}_\alpha(B[m]) \leqslant \mathcal{F}_\alpha(E)$$

with equality if and only if $|E \Delta B[m]| = 0$ up to translation. If instead $m > m_*(\alpha)$ then there exists a sequence of sets of finite perimeter $\{E_i\}_{i=1}^\infty$ with $|E_i| = m$ and $|E_i \Delta B[m]| \to 0$ but $\mathcal{F}_\alpha(B[m]) > \mathcal{F}_\alpha(E_i)$ for every $i \in \mathbb{N}$.

Remark 1.6 (Relationship between variants of $\mathcal{F}_\alpha$). One could consider the a priori more general three-parameter family of functionals

$$\mathcal{E}_{\alpha,\beta}(E) := \mathcal{P}(E) + \beta\,\mathcal{N}_\alpha(E) \qquad \text{over } |E| = m, \qquad (\alpha, \beta, m) \in [0,\infty) \times [0,\infty) \times (0,\infty),$$

where $\mathcal{N}_\alpha(E)$ is defined as in (1.1). Here we are studying $\mathcal{F}_\alpha = \mathcal{E}_{\alpha,1}$ over volume m, whereas in the literature authors sometimes consider $\mathcal{E}_{\alpha,\beta}$ over volume $|B_1|$ instead. Under a volume scaling, studying volume-constrained minimizers of any of these two functionals is equivalent to just studying $\mathcal{F}_\alpha$ for a particular choice of α and m. Indeed, by a change of variables

$$\mathcal{N}_\alpha(tE) = \int_{tE}\int_{tE} Y_\alpha(x-y)\,\mathrm{d}x\,\mathrm{d}y = t^6 \int_E\int_E Y_\alpha(t(x-y))\,\mathrm{d}x\,\mathrm{d}y = t^5 \mathcal{N}_{t\alpha}(E)$$

and combined with the usual scaling of the perimeter yields

$$\mathcal{E}_{\alpha,\beta}(tE) = t^2\left[\mathcal{P}(E) + \beta t^3 \mathcal{N}_{t\alpha}(E)\right]; \tag{1.3}$$

hence, $(\alpha, \beta, m) \mapsto (t\alpha,\ \beta t^3,\ m/t^3)$. For example, minimizers of $\mathcal{E}_{\alpha,\beta}$ with volume m, provided they exist, are (up to scaling) minimizers of $\mathcal{F}_{\alpha\beta^{-1/3}}$ with volume $m\beta$. At times we may switch to studying $\mathcal{E}_{\alpha,\beta}$ for the convenience of presentation. Indeed, Theorem 1.4 and Theorem 1.5 are more easily stated and proven in terms of $\mathcal{E}_{\alpha,\beta}$; see Theorem 5.1 and Theorem 6.1 for the analogous statements, including an explicit stability threshold in terms of β, see (5.1).

We state all the results in the introduction in terms of $\mathcal{F}_\alpha$ to not burden the reader and to provide a physically relevant functional.

Remark 1.7 (Comparison with previous literature). At $\alpha = 0$ the value of $m_*(0)$ is not new. Bonacini and Cristoferi [BC14] and later Figalli, Fusco, Maggi, Millot, and Morini [FFM+15] obtain the unscreened counterpart of Theorem 1.5, i.e. the sharp threshold for local minimality of balls for the perimeter plus a Riesz interaction. In $\mathbb{R}^3$ our $m_*(\alpha)$ agrees with it as $\alpha \to 0$.

In [Peg21], Pegon considers the functionals

$$\mathcal{F}_{\gamma,G_\lambda}(E) := \mathcal{P}(E) - \gamma\lambda^4 \int_E \int_{E^c} G\big(\lambda(x-y)\big)\,\mathrm{d}x\,\mathrm{d}y$$

for some kernel G. In particular, if $G(x) = e^{-|x|}/(2\pi|x|)$ then Pegon proves in [Peg21, Theorem C] that

- if $\gamma < 1$ there exists $\lambda_s(\gamma) > 0$ such that if $\lambda > \lambda_s$, then B_1 is a volume-constrained stable critical point of $\mathcal{F}_{\gamma,G_\lambda}$;
- if $\gamma > 1$ there exists $\lambda_u(\gamma) > 0$ such that if $\lambda > \lambda_u$, then B_1 is a volume-constrained unstable critical point of $\mathcal{F}_{\gamma,G_\lambda}$.

Using that $\|Y_\lambda\|_{L^1(\mathbb{R}^3)} = 4\pi/\lambda^2$, we have

$$\mathcal{F}_{\gamma,G_\lambda}(E) = \mathcal{P}(E) + \gamma\lambda^3 \int_E \int_E \frac{e^{-\lambda|x-y|}}{2\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y - 2\gamma\lambda|E| = \mathcal{E}_{\lambda,\gamma\lambda^3/2\pi}(E) - 2\gamma\lambda|E|.$$

Hence, analyzing volume-constrained stable critical points E with $|E| = |B_1|$ of $\mathcal{F}_{\gamma,G_\lambda}$ is equivalent to that of $\mathcal{F}_\alpha$ over E with $|E| = m$, where $\alpha = (2\pi/\gamma)^{1/3}$ and $m = (\gamma\lambda^3/2\pi)|B_1|$. In particular, when $\gamma > 1$ we have $\alpha < (2\pi)^{1/3}$, and so by Theorem 1.4 we can recover Pegon's result for some explicitly computable λ_u, and moreover show that if $\lambda \leqslant \lambda_u$ then B_1 is a volume-constrained stable critical point of $\mathcal{F}_{\gamma,G_\lambda}$. On the other hand if $\gamma \leqslant 1$ then $\alpha \geqslant (2\pi)^{1/3}$, and so B_1 is *always* a volume-constrained stable critical point; hence $\lambda_s = 0$.

Remark 1.8. One could also study an analogue of $\mathcal{F}_\alpha$ in higher dimensions, whose kernel arises as the fundamental solution of $-\Delta + \alpha^2$. For the sake of presentation we elect to focus on dimension $n = 3$ as the kernel is well studied in that case, see [WDNWB21]. However, we believe that similar results hold in any dimension.

Acknowledgments. LB is supported by an NSERC discovery grant. KDM was supported by NSF-DMS 2306962. This work was partially completed while KDM was hosted at the Department of Mathematics and Statistics of Virginia Commonwealth University. IT was partially supported by the Simons Foundation MPS-TSM 851065, and by NSF-DMS 2306962 and NSF-DMS 2607148. The authors would also like to acknowledge support from the Fields Institute through their Fields Opportunity for Collaborations US (FOCUS) program.

2. Notation and Preliminaries

As in [Peg21], we will denote by $\mathcal{P}_\alpha$ the nonlocal functional

$$\mathcal{P}_\alpha(E) := \int_E \int_{E^c} Y_\alpha(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Since $\|Y_\alpha\|_{L^1(\mathbb{R}^3)} = 4\pi/\alpha^2$, recalling (1.1), we have $\mathcal{N}_\alpha(E) = 4\pi|E|/\alpha^2 - \mathcal{P}_\alpha(E)$, so $\mathcal{F}_\alpha(E) = \mathcal{G}_\alpha(E) + 4\pi|E|/\alpha^2$, where

$$\mathcal{G}_\alpha(E) := \mathcal{P}(E) - \mathcal{P}_\alpha(E).$$

As a consequence,

$$g_\alpha(m) := \inf\left\{\mathcal{G}_\alpha(E) \colon |E| = m\right\} = e_\alpha(m) - \frac{4\pi m}{\alpha^2},$$

where $e_\alpha(m)$ is defined in (1.2).

Throughout the paper we will refer to the following constants related to the surface tension:

$$\sigma_\alpha := \frac{2\pi}{\alpha^3}, \qquad \theta_\alpha := 1 - \sigma_\alpha,$$

as well as the following ones related to the local and nonlocal terms of the energy:

$$C_1 := (36\pi)^{1/3}, \qquad C_2 := \frac{32\pi^2}{15}\left(\frac{3}{4\pi}\right)^{5/3}, \qquad \text{so that} \quad \overline{m} = \frac{2^{1/3}-1}{1-2^{-2/3}}\frac{C_1}{C_2}, \tag{2.1}$$

where $\overline{m}$ is defined in the statement of Theorem 1.2. Here C_1 and C_2 are such that $\mathcal{P}(E) \geqslant C_1|E|^{2/3}$, and $\mathcal{N}_0(B[m]) = C_2 m^{5/3}$, where $B[m]$ denotes the ball in $\mathbb{R}^3$ of volume m. Also note that $\theta_\alpha > 0$ simply means $\alpha > (2\pi)^{1/3}$.

In the following lemma we collect the properties of the Yukawa kernel that we will need in the paper.

Lemma 2.1 (Properties of Y_α)**.** *For any $\alpha \geqslant 0$, the function Y_α satisfies the following.*

(i) *Y_α is radial, strictly decreasing in $|z|$, and $0 < Y_\alpha \leqslant Y_0 = 1/|z|$.*

(ii) $|\nabla Y_\alpha(z)| = (1+\alpha|z|)e^{-\alpha|z|}/|z|^2 \leqslant 1/|z|^2$.

(iii) *Y_α is positive definite. Namely $\widehat{Y_\alpha}(k) = 4\pi/(\alpha^2+|k|^2) > 0$, and so*

$$\iint Y_\alpha(x-y)f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y \geqslant 0$$

for any $f \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with compact support.

(iv) *For any $E \subset \mathbb{R}^3$ with $|E| \leqslant |B_1|$, and for any $x \in \mathbb{R}^3$, we have*

$$\int_E Y_\alpha(x-y)\,\mathrm{d}y \leqslant \int_{B_1} \frac{1}{|y|}\,\mathrm{d}y = 2\pi,$$

$$\int_E |\nabla Y_\alpha(x-y)|\,\mathrm{d}y \leqslant \int_{B_1} \frac{1}{|y|^2}\,\mathrm{d}y = 4\pi.$$

(v) *For any $E \subset \mathbb{R}^3$ we have $\mathcal{N}_\alpha(E) \leqslant \mathcal{N}_\alpha(B[|E|]) \leqslant \mathcal{N}_0(B[|E|])$. In particular, $\mathcal{N}_\alpha(E) \leqslant C_2|E|^{5/3}$ and $\mathcal{N}_\alpha(B_1) \leqslant 32\pi^2/15$.*

Proof. The first item follows from the definition of Y_α. For the second item we use $(1+t)e^{-t} \leqslant 1$. The third one follows from the fact that $(-\Delta + \alpha^2)Y_\alpha = 4\pi\delta_0$. Items (iv) and (v) use the fact that $Y_\alpha \leqslant Y_0$ combined with the Riesz rearrangement inequality. □

Next we collect some additional properties about the Yukawa energy of B_1.

Lemma 2.2 (Properties of $\mathcal{N}_\alpha(B_1)$)**.** *For any $\alpha \geqslant 0$ the following hold:*

(i) *$\mathcal{N}_\alpha(B_1)$ is explicitly given by*

$$\mathcal{N}_\alpha(B_1) = \frac{8\pi^2 e^{-\alpha}}{3\alpha^5}\Big(\left(2\alpha^3 - 3\alpha^2 + 3\right) e^{\alpha} - 3(\alpha+1)^2 e^{-\alpha}\Big).$$

(ii) *Denoting $\mathcal{N}_\alpha^{(k)}(B_1) = \mathrm{d}^k/\mathrm{d}\alpha^k\, \mathcal{N}_\alpha(B_1)$, there exists a $t_0 > 0$, independent of α, such that if $t \in (-t_0, t_0)$ then*

$$\mathcal{N}_\alpha(B_1) + \alpha\mathcal{N}_\alpha'(B_1)\, t + \alpha^3 \mathcal{N}_\alpha'''(B_1)\, \frac{t^3}{6} \geqslant 0.$$

(iii) *The following inequalities hold*

$$\mathcal{N}_\alpha''(B_1) > 0, \qquad 50\mathcal{N}_\alpha(B_1) + 58\alpha\mathcal{N}_\alpha'(B_1) + 15\alpha^2\mathcal{N}_\alpha''(B_1) + \alpha^3\mathcal{N}_\alpha'''(B_1) > 0,$$
$$60\mathcal{N}_\alpha'(B_1) + 30\alpha\mathcal{N}_\alpha''(B_1) + 5\alpha^2\mathcal{N}_\alpha'''(B_1) < 0, \qquad \mathcal{N}_\alpha'''(B_1) < 0.$$

Proof. Without loss of generality assume that $y = (0,0,\rho)$. Then in spherical coordinates

$$\begin{aligned}\int_{B_1} \frac{e^{-\alpha|x-y|}}{|x-y|}\,\mathrm{d}x &= 2\pi \int_0^1 \int_0^\pi \frac{e^{-\alpha\sqrt{\rho^2+r^2-2r\rho\cos(\varphi)}}}{\sqrt{\rho^2+r^2-2r\rho\cos(\varphi)}}\, r^2 \sin(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}r \\ &= \frac{2\pi}{\rho}\int_0^1 r \int_{|\rho-r|}^{\rho+r} e^{-\alpha u}\,\mathrm{d}u\,\mathrm{d}r = \frac{2\pi}{\alpha\rho}\int_0^1 r(e^{-\alpha|\rho-r|} - e^{-\alpha(\rho+r)})\,\mathrm{d}r = f(\rho),\end{aligned}$$

where

$$f(\rho) \coloneqq \frac{2\pi}{\alpha^3\rho}\Big((\alpha+1)e^{-\alpha(1+\rho)} - (\alpha+1)e^{-\alpha(1-\rho)} + 2\alpha\rho\Big).$$

Consequently, we have that $\mathcal{N}_\alpha(B_1)$ is given by

$$\begin{aligned}\mathcal{N}_\alpha(B_1) &= \int_{B_1}\int_{B_1} \frac{e^{-\alpha|x-y|}}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y = 4\pi\int_0^1 \rho^2 f(\rho)\,\mathrm{d}\rho \\ &= \frac{8\pi^2 e^{-\alpha}}{3\alpha^5}\Big(\left(2\alpha^3 - 3\alpha^2 + 3\right) e^{\alpha} - 3(\alpha+1)^2 e^{-\alpha}\Big).\end{aligned}$$

Items (ii) and (iii) then follow by direct computation. □

Defining the potential of the unit ball as

$$v_\alpha(x) \coloneqq \int_{B_1} Y_\alpha(x-y)\,\mathrm{d}y$$

we obtain the following lemma that will be used in the sequel as well. The radially symmetric decreasing property of v_α is a simple consequence of the layer cake representation (see for example [LL01, Theorem 1.13]) and the bound on the Lipschitz constant follows from an explicit calculation.

Lemma 2.3. *For any $\alpha \geqslant 0$, the potential v_α is radial and radially decreasing. Moreover* $\mathrm{Lip}(v_\alpha) \leqslant 4\pi$.

3. Existence of minimizers of $\mathcal{F}_\alpha$

In this section we prove Theorems 1.1 and 1.2 following [MNS25]. The main tool in their approach is the boundary representation of the energy $\mathcal{F}_\alpha$, which isolates the effective tension and a nonnegative defect. For this, we define

$$\psi_\alpha(r) := \frac{(1+\alpha r)e^{-\alpha r}}{\alpha^2 r^2},$$

and for any $y \in \partial^* E$, we let

$$H_y := \big\{x \colon \nu_E(y)\cdot(x-y) < 0\big\}$$

be the inner-tangent half space, where ν_E denotes the outer unit normal to E.

Proposition 3.1 (Boundary representation, cf. [MNS25, Lemma 3.1]). *For any $\alpha > 0$, we have*

$$\mathcal{P}_\alpha(E) = \int_{\partial^* E}\int_E \psi_\alpha(|x-y|)\nu_E(y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y) = \sigma_\alpha \mathcal{P}(E) - \mathcal{D}_\alpha(E),$$

where

$$\mathcal{D}_\alpha(E) := \int_{\partial^* E}\int_{E\triangle H_y} \psi_\alpha(|x-y|)\left|\nu_E(y)\cdot\frac{y-x}{|y-x|}\right|\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y) \geqslant 0.$$

Consequently, $\mathcal{P}_\alpha(E) \leqslant \sigma_\alpha \mathcal{P}(E)$ and $\mathcal{F}_\alpha(E) = \theta_\alpha \mathcal{P}(E) + \mathcal{D}_\alpha(E) + 4\pi|E|/\alpha^2$.

Proof. Let $\Phi_\alpha = Y_\alpha/\alpha^2 = e^{-\alpha|z|}/(\alpha^2|z|)$. Then $\nabla\Phi_\alpha(z) = -\psi_\alpha(|z|)\,z/|z|$, and

$$\Delta\Phi_\alpha = \frac{\Delta Y_\alpha}{\alpha^2} = Y_\alpha \qquad \text{on } \mathbb{R}^3\backslash\{0\}.$$

Now fix a point $x \in E$ of density one, and consider the function $y \mapsto \nabla_y \Phi_\alpha(x-y)$. Choose $\theta, \eta \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} &\theta = 0 \text{ on } (-\infty,1], \quad \theta = 1 \text{ on } [2,\infty), \quad 0 \leqslant \theta \leqslant 1,\\ &\eta = 1 \text{ on } (-\infty,1], \quad \eta = 0 \text{ on } [2,\infty), \quad 0 \leqslant \eta \leqslant 1. \end{aligned}$$

For $0 < \varepsilon < R$, define $\Xi_{\varepsilon,R}(y) := \theta(|x-y|/\varepsilon)\,\eta(|x-y|/R)$. Then $\Xi_{\varepsilon,R} \equiv 0$ for $|x-y| \leqslant \varepsilon$ and for $|x-y| \geqslant 2R$; hence, the singularities at $y = x$ and $y = \infty$ are removed, and

$$V_{\varepsilon,R}(y) := \Xi_{\varepsilon,R}(y)\nabla_y\Phi_\alpha(x-y) \in C^1_c(\mathbb{R}^3).$$

By the divergence theorem, applied in E^c, we get

$$\begin{aligned} -\int_{\partial^* E} \Xi_{\varepsilon,R}\nabla_y\Phi_\alpha(x-y)\cdot\nu_E(y)\,\mathrm{d}\mathcal{H}^2(y) &= \int_{\partial^* E^c} V_{\varepsilon,R}(y)\cdot\nu_{E^c}(y)\,\mathrm{d}\mathcal{H}^2(y)\\ &= \int_{E^c} \operatorname{div}_y V_{\varepsilon,R}(y)\,\mathrm{d}y = I_1 + I_2, \end{aligned}$$

where

$$I_1 := \int_{E^c} \Xi_{\varepsilon,R} Y_\alpha(x-y)\,\mathrm{d}y \qquad \text{and} \qquad I_2 := \int_{E^c} \nabla_y \Xi_{\varepsilon,R}(y)\cdot\nabla_y\Phi_\alpha(x-y)\,\mathrm{d}y.$$

The dominated convergence theorem implies that

$$I_1 \to \int_{E^c} Y_\alpha(x-y)\,\mathrm{d}y \qquad \text{as } \varepsilon \to 0 \text{ and } R \to \infty.$$

In order to estimate I_2 we will write it as a sum of an inner shell and an outer shell. Precisely,

$$\begin{aligned} I_2 &= I_{2,\mathrm{inn}} + I_{2,\mathrm{out}} \\ &:= \int_{E^c} \eta\left(\frac{|x-y|}{R}\right) \nabla_y \left[\theta\left(\frac{|x-y|}{\varepsilon}\right)\right] \cdot \nabla_y \Phi_\alpha(x-y)\,\mathrm{d}y \\ &\qquad + \int_{E^c} \theta\left(\frac{|x-y|}{\varepsilon}\right) \nabla_y \left[\eta\left(\frac{|x-y|}{R}\right)\right] \cdot \nabla_y \Phi_\alpha(x-y)\,\mathrm{d}y. \end{aligned}$$

The outer shell is supported on $\{R \leqslant |x-y| \leqslant 2R\}$ with $|\nabla_y \eta| \leqslant C/R$. On the support,

$$|\nabla_y \Phi_\alpha(x-y)| = \psi_\alpha(|x-y|) \leqslant \frac{(1+2\alpha R)e^{-\alpha R}}{\alpha^2 R^2} \leqslant \frac{Ce^{-\alpha R}}{\alpha R},$$

and the shell has volume $O(R^3)$. Therefore

$$|I_{2,\mathrm{out}}| \leqslant \frac{C}{R}\frac{e^{-\alpha R}}{\alpha R} R^3 \to 0 \qquad \text{as } R \to \infty.$$

The inner shell, on the other hand, is supported on $\{\varepsilon \leqslant |x-y| \leqslant 2\varepsilon\}$ with $|\nabla_y \theta| \leqslant C/\varepsilon$. On the support,

$$|\nabla_y \Phi_\alpha(x-y)| = \psi_\alpha(|x-y|) \leqslant \frac{1}{\alpha^2 \varepsilon^2}.$$

Thus

$$|I_{2,\mathrm{inn}}| \leqslant \frac{C}{\varepsilon}\frac{1}{\alpha^2\varepsilon^2}|E^c \cap B_{2\varepsilon}(x)| \to 0 \qquad \text{as } \varepsilon \to 0,$$

since $|E^c \cap B_{2\varepsilon}(x)| = o(\varepsilon^3)$ due to the fact that x is a point of density one. Putting these together yields $I_2 \to 0$ as $\varepsilon \to 0$ and $R \to \infty$.

Next, we will consider the boundary integral. First note that for almost every $x \in E$, $\int_{\partial^* E} \psi_\alpha(|x-y|)\,\mathrm{d}\mathcal{H}^2(y)$ is finite, since, by symmetric rearrangement,

$$\int_E \int_{\partial^* E} \psi_\alpha(|x-y|)\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x = \int_{\partial^* E}\int_E \psi_\alpha(|x-y|)\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y) \leqslant \frac{4\pi\rho}{\alpha^2}\mathcal{P}(E),$$

where ρ is such that $|B_\rho| = |E|$. Then, by the dominated convergence theorem,

$$\int_{\partial^* E} \Xi_{\varepsilon,R} \nabla_y \Phi_\alpha(x-y) \cdot \nu_E(y)\,\mathrm{d}\mathcal{H}^2(y) \to \int_{\partial^* E} \nabla_y \Phi_\alpha(x-y) \cdot \nu_E(y)\,\mathrm{d}\mathcal{H}^2(y),$$

as $\varepsilon \to 0$ and $R \to \infty$. Therefore,

$$\int_{E^c} Y_\alpha(x-y)\,\mathrm{d}y = -\int_{\partial^* E} \nabla_y \Phi_\alpha(x-y)\cdot\nu_E(y)\,\mathrm{d}\mathcal{H}^2(y) = \int_{\partial^* E} \psi_\alpha(|x-y|)\,\nu_E(y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}\mathcal{H}^2(y),$$

and integrating over E and using Fubini's theorem yields

$$\mathcal{P}_\alpha(E) = \int_{\partial^* E}\int_E \psi_\alpha(|x-y|)\,\nu_E(y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y).$$

Fixing $y \in \partial^* E$, and letting $w = x-y$, we have that $x \in H_y$ if and only if $\nu_E \cdot w < 0$. Hence,

$$\begin{aligned} \int_{H_y} \psi_\alpha(|x-y|)\nu_E(y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}x &= \int_{\{\nu_E\cdot w<0\}} \psi_\alpha(|w|)\left|\nu_E \cdot \frac{w}{|w|}\right|\,\mathrm{d}w \\ &= \int_0^\infty \psi_\alpha(r) r^2\,\mathrm{d}r \int_{\{\nu_E\cdot\omega<0\}} |\nu_E\cdot\omega|\,\mathrm{d}\mathcal{H}^2(\omega) = \frac{2\pi}{\alpha^3} = \sigma_\alpha, \end{aligned}$$

which is independent of y. Now, note that on $E \setminus H_y$ we have $\nu_E \cdot (y-x) \leqslant 0$, so

$$\psi_\alpha(|x-y|)\nu_E(y) \cdot \frac{y-x}{|y-x|} \leqslant 0,$$

and reverse inequalities hold on $H_y \setminus E$. Thus, writing $E = H_y \cup (H_y \setminus E) \cup (E \setminus H_y)$, we obtain

$$\int_E \psi_\alpha(|x-y|)\nu_E(y) \cdot \frac{y-x}{|y-x|}\,\mathrm{d}x = \sigma_\alpha - \int_{E \triangle H_y} \psi_\alpha(|x-y|) \left| \nu_E(y) \cdot \frac{y-x}{|y-x|} \right| \mathrm{d}x.$$

Finally, integrating over $\partial^* E$ we get $\mathcal{P}_\alpha(E) = \sigma_\alpha \mathcal{P}(E) - \mathcal{D}_\alpha(E)$. □

The next lemma gives the scaling and monotonicity properties of $\mathcal{D}_\alpha$. It follows easily from the scaling properties of the perimeter, the kernel Y_α, and the functional $\mathcal{P}_\alpha$ combined with the monotonicity of the function $t \mapsto (1+t)e^{-t}$.

Lemma 3.2. *For any $t > 0$, $\mathcal{D}_\alpha(tE) = t^5 \mathcal{D}_{t\alpha}(E)$. Moreover, for any $0 < \lambda \leqslant 1$ and $r > 0$, $\lambda^2 \psi_{\lambda\alpha}(r) \geqslant \psi_\alpha(r)$, and $\lambda^2 \mathcal{D}_{\lambda\alpha}(E) \geqslant \mathcal{D}_\alpha(E)$ for any $E \subset \mathbb{R}^3$.*

Finally, the core of the proof of Theorem 1.1 relies on the following compactness result which is a classical consequence of strict subadditivity, and can be obtained by arguing as in [FL15, Proposition 2.1, Lemma 2.2, Theorem 3.1] or [NP21, Proposition 1.2, Lemma 3.4] combined with the continuity of $\mathcal{N}_\alpha$.

Proposition 3.3. *For any $\alpha \geqslant 0$ and $m > 0$, if*

$$e_\alpha(m) < e_\alpha(\xi) + e_\alpha(m-\xi) \qquad \text{for all } \xi \in (0,m),$$

then $e_\alpha(m)$ is attained.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. In light of Proposition 3.1 we have $\mathcal{G}_\alpha = \theta_\alpha \mathcal{P} + \mathcal{D}_\alpha$. By Lemma 3.2, we have that

$$\mathcal{D}_\alpha(t^{1/3}E) = t^{5/3} \mathcal{D}_{t^{1/3}\alpha}(E) \geqslant t\mathcal{D}_\alpha(E).$$

Adding $\theta_\alpha t^{2/3} \mathcal{P}(E)$ to both sides then yields

$$\begin{aligned} \mathcal{G}_\alpha(t^{1/3}E) &= \theta_\alpha \mathcal{P}(t^{1/3}E) + \mathcal{D}_\alpha(t^{1/3}E) = \theta_\alpha t^{2/3}\mathcal{P}(E) + \mathcal{D}_\alpha(t^{1/3}E) \\ &\geqslant \theta_\alpha (t + \varphi(t))\mathcal{P}(E) + t\mathcal{D}_\alpha(E) = t\mathcal{G}_\alpha(E) + \theta_\alpha \varphi(t)\mathcal{P}(E), \end{aligned}$$

where $\varphi(t) := t^{2/3} - t > 0$ on $(0,1)$. Now, letting $t = m'/m$ for $0 < m' < m$ and taking the infimum of both sides over sets of volume m we obtain

$$g_\alpha(m') \geqslant \frac{m'}{m} g_\alpha(m) + \theta_\alpha C_1 m^{2/3} \varphi\left(\frac{m'}{m}\right), \tag{3.1}$$

where we also used the inequality $\mathcal{P}(E) \geqslant C_1 m^{2/3}$. Adding $4\pi m'/\alpha^2$ to both sides, and using the fact that $\varphi(m'/m) > 0$ we get that

$$\frac{e_\alpha(m')}{m'} > \frac{e_\alpha(m)}{m};$$

hence, $m^{-1} e_\alpha(m)$ is strictly decreasing.

For $0 < \xi < m$, we can apply the inequality (3.1) with $m' = \xi$ and $m' = m - \xi$. Adding these two inequalities we obtain

$$e_\alpha(\xi) + e_\alpha(m-\xi) \geqslant e_\alpha(m) + \theta_\alpha C_1 m^{2/3}\left[\varphi\left(\frac{\xi}{m}\right) + \varphi\left(\frac{m-\xi}{m}\right)\right];$$

hence, again thanks to the positivity of φ, we get the strict binding inequality

$$e_\alpha(m) < e_\alpha(\xi) + e_\alpha(m-\xi),$$

and the result follows from Proposition 3.3. □

Remark 3.4. The only estimates used to prove the strict binding inequality above come from Lemma 3.2 and the isoperimetric inequality. By quantifying these, one could potentially prove a stronger strict binding inequality. For instance, we have that

$$\lambda^2 \psi_{\lambda\alpha}(r) - \psi_\alpha(r) = \frac{1}{\alpha^2 r^2}\int_{\lambda\alpha r}^{\alpha r} t e^{-t}\,\mathrm{d}t \geqslant \frac{1-\lambda^2}{2} e^{-\alpha r},$$

which leads to the corresponding estimate for $\mathcal{D}_\alpha$ as

$$\lambda^2 \mathcal{D}_{\lambda\alpha}(E) \geqslant \mathcal{D}_\alpha(E) + \frac{1-\lambda^2}{2}\int_{\partial^* E}\int_{E\Delta H_y} e^{-\alpha|x-y|}\left|\nu_E(y)\cdot\frac{x-y}{|x-y|}\right|\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y).$$

If we naïvely use the argument of Proposition 3.1 in reverse to rewrite the latter integral, then

$$\int_{\partial^* E}\int_{E\Delta H_y} e^{-\alpha|x-y|}\left|\nu_E(y)\cdot\frac{x-y}{|x-y|}\right|\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y) = \sigma_\alpha \mathcal{P}(E) - \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\alpha^2 \mathcal{N}_\alpha(E)\right);$$

however, it is unclear if this gives a better result.

Now we turn to the proof of Theorem 1.2. For this we will follow the arguments in [FN21] and utilize the following lemma, whose proof requires a minor modification of [FN21, Lemma 5].

Lemma 3.5. *For any $\alpha \geqslant 0$ and $0 < m' < m$, if $t = m'/m$, then*

$$e_\alpha(m') \geqslant t^{5/3} e_\alpha(m) + (1-t)t^{2/3} C_1 m^{2/3}. \tag{3.2}$$

Proof. Let $E \subset \mathbb{R}^3$ be a set such that $|E| = m'$. Then $|t^{-1/3}E| = m$, and $\mathcal{P}(t^{-1/3}E) = t^{-2/3}\mathcal{P}(E)$, and

$$\mathcal{N}_\alpha(t^{-1/3}E) = t^{-5/3}\mathcal{N}_{t^{-1/3}\alpha}(E) \leqslant t^{-5/3}\mathcal{N}_\alpha(E),$$

since $t^{-1/3}\alpha \geqslant \alpha$ (for $\alpha = 0$ we have an equality above and recover the homogeneous case of [FN21]). Therefore

$$e_\alpha(m) \leqslant \mathcal{F}_\alpha(t^{-1/3}E) = t^{-5/3}\mathcal{F}_\alpha(E) - \left(t^{-5/3} - t^{-2/3}\right)\mathcal{P}(E).$$

Since $t < 1$ then $t^{-5/3} - t^{-2/3} > 0$, and by the isoperimetric inequality, $\mathcal{P}(E) \geqslant C_1 t^{2/3} m^{2/3}$; hence,

$$e_\alpha(m) \leqslant t^{-5/3}\mathcal{F}_\alpha(E) - \left(t^{-5/3} - t^{-2/3}\right)t^{2/3} C_1 m^{2/3}.$$

Taking the infimum over sets of volume m' and multiplying both sides by $t^{5/3}$ yields the inequality (3.2). □

We now prove Theorem 1.2 following [FN21, Theorem 4].

Proof of Theorem 1.2. We apply the previous lemma twice, with $m' = \xi$ and $m' = m - \xi$. Then adding both inequalities and letting $t = \xi/m$, we obtain

$$\begin{aligned} &e_\alpha(\xi) + e_\alpha(m-\xi) - e_\alpha(m) \\ &\qquad\qquad \geqslant \big(t^{5/3} + (1-t)^{5/3} - 1\big)e_\alpha(m) + \big((1-t)t^{2/3} + t(1-t)^{2/3}\big)C_1 m^{2/3}. \end{aligned}$$

Since $t^{5/3} + (1-t)^{5/3} - 1 < 0$, by using the ball of volume m as a competitor,

$$\begin{aligned} e_\alpha(\xi) + e_\alpha(m-\xi) - e_\alpha(m) &\geqslant C_1 m^{2/3}\Big[\big(t^{5/3} + (1-t)^{5/3} - 1\big) + \big((1-t)t^{2/3} + t(1-t)^{2/3}\big)\Big] \\ &\qquad + \mathcal{N}_\alpha(B[m])\big(t^{5/3} + (1-t)^{5/3} - 1\big) \\ &= C_1\big(t^{2/3} + (1-t)^{2/3} - 1\big)m^{2/3} + \big(t^{5/3} + (1-t)^{5/3} - 1\big)\mathcal{N}_\alpha(B[m]). \end{aligned}$$

Since $\mathcal{N}_\alpha(B[m]) \leqslant \mathcal{N}_0(B[m]) = C_2 m^{5/3}$ and, by [FN21, Eq. (9)],

$$\inf_{t\in(0,1)} \frac{t^{2/3} + (1-t)^{2/3} - 1}{1 - t^{5/3} - (1-t)^{5/3}} = \frac{2^{1/3} - 1}{1 - 2^{-2/3}},$$

we get that the right-hand side of the inequality above is positive for $m < \overline{m}$ with $\overline{m}$ defined in (2.1). Thus we have the strict binding inequality

$$e_\alpha(m) < e_\alpha(\xi) + e_\alpha(m-\xi),$$

which, in turn, implies the theorem via Proposition 3.3. □

Remark 3.6 (Optimality of $\overline{m}$). The threshold $\overline{m}$ is exactly the volume at which a ball has the same energy for $\alpha = 0$ as two balls of half the volume placed infinitely far apart. This is a ceiling for any strict-binding argument: beyond it, if balls are minimizers then $e_0(m) > 2e_0(m/2)$ and strict subadditivity is false. In [FN21], where the nonlocal term has a factor of $1/2$, the authors obtain the same threshold $m_* = 2\overline{m}$.

Remark 3.7 (An α-dependent refinement). The kernel enters Theorem 1.2 only through the ball competitor, and with its closed form $\mathcal{N}_\alpha(B_R) = 16\pi^2/\alpha^5\, \Psi(\alpha R)$, where $\Psi(u) = u^3/3 - (1+u)(u\cosh u - \sinh u)e^{-u}$, it yields thresholds $m(\alpha)$ strictly larger than $\overline{m}$ for every $\alpha > 0$. However $\mathcal{N}_\alpha(B[m])$ decreases in α, so the infimum over α is achieved when $\alpha = 0$ and a threshold uniform in α cannot be improved this way. Moreover $m(\alpha)$ is increasing, whereas for $\sigma_\alpha < 1$ existence already holds at every volume by Theorem 1.1.

4. Minimality of balls for $\mathcal{F}_\alpha$

In this section we prove Theorem 1.3. We first observe owing to Remark 1.6 that we can instead consider the functional

$$\mathcal{E}_{\alpha,\beta}(E) = \mathcal{P}(E) + \beta\mathcal{N}_\alpha(E)$$

for $E \subset \mathbb{R}^3$ with $|E| = |B_1|$. With this reformulation, Theorem 1.3 follows from

Theorem 4.1. *There exists $\beta_0 > 0$ (independent of α) such that for any $\alpha \geqslant 0$ and $\beta < \beta_0$, the unique minimizer, up to translations, of $\mathcal{E}_{\alpha,\beta}$ over sets $|E| = |B_1|$ is the unit ball B_1.*

Proof of Theorem 1.3. For $m > 0$ and $\alpha \geqslant 0$, by scaling, see (1.3), we have that $B[m]$ minimizes $\mathcal{F}_\alpha(E)$ among sets $|E| = m$ if and only if B_1 minimizes $\mathcal{E}_{\alpha',\beta}(E)$ among sets $|E| = |B_1|$, where $\alpha' = \alpha\beta^{1/3}$ with $\beta = m/|B_1|$. Hence taking $m_0 = \beta_0|B_1|$ we have $\beta < \beta_0$ whenever $m < m_0$, and Theorem 4.1 gives the result. □

We prove Theorem 4.1 following the arguments in [CFP23]. The main tool in establishing the theorem is the following lemma which improves the deficit of the nonlocal term when the set E is nearly spherical. While in [CFP23] the authors prove a similar result by explicitly controlling the interactions between $E\setminus B_1$ and $B_1\setminus E$, here the positive definiteness of Y_α simplifies the proof.

Lemma 4.2. *Let $\alpha \geqslant 0$ and suppose E is a nearly spherical set defined as*

$$E = \big\{tx \in \mathbb{R}^3 \colon x \in \partial B_1,\ 0 \leqslant t < 1 + u(x)\big\}$$

for some $u \in C^1(\partial B_1)$ such that $|E| = |B_1|$ and $\|u\|_{L^\infty(\partial B_1)} \leqslant 1/2$. Then

$$\mathcal{N}_\alpha(B_1) - \mathcal{N}_\alpha(E) \leqslant 9\pi \|u\|^2_{L^2(\partial B_1)}.$$

Proof. Note that the function $\chi_E - \chi_{B_1} \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and has compact support. Hence, by Lemma 2.1(iii),

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} Y_\alpha(x-y)\big(\chi_E - \chi_{B_1}\big)(x)\big(\chi_E - \chi_{B_1}\big)(y)\,\mathrm{d}x\,\mathrm{d}y \geqslant 0.$$

Expanding the energy of E, this implies that

$$\mathcal{N}_\alpha(B_1) - \mathcal{N}_\alpha(E) \leqslant -2\int_{\mathbb{R}^3} \big(\chi_E - \chi_{B_1}\big)(x) v_\alpha(x)\,\mathrm{d}x.$$

Now, let $\overline{v}_\alpha := v_\alpha(x_0)$ for any $x_0 \in \partial B_1$, which is a constant due to Lemma 2.3. Since $\int_{\mathbb{R}^3}(\chi_E - \chi_{B_1})(x)\,\mathrm{d}x = 0$ and $|\chi_E - \chi_{B_1}| \leqslant 1$ with $\chi_E - \chi_{B_1}$ supported on $E \triangle B_1$, by triangle inequality,

$$\begin{aligned} &-2\int_{\mathbb{R}^3} \big(\chi_E - \chi_{B_1}\big)(x) v_\alpha(x)\,\mathrm{d}x \\ &\qquad\qquad = -2\int_{\mathbb{R}^3} \big(\chi_E - \chi_{B_1}\big)(x)(v_\alpha(x) - \overline{v}_\alpha)\,\mathrm{d}x \leqslant 2\int_{E\triangle B_1} |v_\alpha(x) - \overline{v}_\alpha|\,\mathrm{d}x. \end{aligned}$$

This, again using Lemma 2.3, implies that

$$\mathcal{N}_\alpha(B_1) - \mathcal{N}_\alpha(E) \leqslant 8\pi \int_{E\triangle B_1} \big||x| - 1\big|\,\mathrm{d}x \leqslant 8\pi \cdot \frac{9}{4}\int_{\partial B_1} \frac{u^2(\theta)}{2}\,\mathrm{d}\theta = 9\pi\|u\|^2_{L^2(\partial B_1)},$$

where the second inequality uses the fact that $(1 + \|u\|_{L^\infty(\partial B_1)})^2 \leqslant 9/4$. $\square$

As a consequence we get the following regularity result, which follows from [Mag12, Chapter 21] combined with arguments in [CFP23, GMP24].

Proposition 4.3. *There exists $\beta_1 > 0$, $\gamma \in (0, 1/2)$, and an increasing function $\eta\colon (0,\beta_1) \to (0,\infty)$ with $\eta(\beta) \to 0$ as $\beta \to 0$, all independent of α, such that if $\alpha \geqslant 0$, $0 < \beta < \beta_1$, and E minimizes $\mathcal{E}_{\alpha,\beta}$ among sets of volume $|B_1|$, then, up to a translation, E is a nearly spherical set with barycenter at the origin and $\|u\|_{C^{1,\gamma}(\partial B_1)} \leqslant \eta(\beta)$.*

Proof of Theorem 1.3. By [Fug89], there exist $\delta > 0$ and $C_F > 0$ such that if E is a nearly spherical set with barycenter at the origin and $\|u\|_{C^1(\partial B_1)} \leqslant \delta$, then

$$\mathcal{P}(E) - \mathcal{P}(B_1) \geqslant C_F\|u\|^2_{H^1(\partial B_1)}.$$

Now, take

$$\beta_0 := \min\Big\{\beta_1, \frac{C_F}{9\pi}, \sup\big\{\beta < \beta_1 \colon \eta(\beta) \leqslant \min(\delta, 1/2)\big\}\Big\}$$

with β_1 and η from the proposition above. Let $\beta < \beta_0$ and let E be a minimizer of $\mathcal{E}_{\alpha,\beta}$. Then E is a nearly spherical set with barycenter at the origin and $\|u\|_{C^1(\partial B_1)} \leqslant \eta(\beta) \leqslant \min\{\delta, 1/2\}$. Using Lemma 4.2,

$$\mathcal{E}_{\alpha,\beta}(E) - \mathcal{E}_{\alpha,\beta}(B_1) = \big(\mathcal{P}(E) - \mathcal{P}(B_1)\big) - \beta\big(\mathcal{N}_\alpha(B_1) - \mathcal{N}_\alpha(E)\big) \geqslant \big(C_F - 9\pi\beta\big)\|u\|^2_{H^1(\partial B_1)}.$$

Since $\beta < C_F/(9\pi)$ the right-hand side is positive unless $u \equiv 0$, i.e. $E = B_1$. □

5. Sharp stability of B_1

In this section we prove Theorem 1.4. As in Section 4, due to Remark 1.6 it suffices to prove

Theorem 5.1. *For any $\alpha \geqslant 0$ the unit ball B_1 is a volume-constrained stable critical point of $\mathcal{E}_{\alpha,\beta}$ if and only if $\beta \leqslant \beta_*(\alpha)$ where*

$$\beta_*(\alpha) = \frac{\alpha^5 e^\alpha}{(2\alpha^2 - 9)\pi e^\alpha + \left((\alpha^2 + 3\alpha + 3)^2 + \alpha^2(\alpha+1)^2\right)\pi e^{-\alpha}}. \tag{5.1}$$

Proof of Theorem 1.4. For $m > 0$ and $\alpha \geqslant 0$, by scaling we have that $B[m]$ is a volume-constrained stable critical point of $\mathcal{F}_\alpha$ if and only if B_1 is a volume-constrained stable critical point of $\mathcal{E}_{\alpha',\beta}$ with $\alpha' = \alpha\beta^{1/3}$ and $\beta = m/|B_1|$. Theorem 5.1 then gives that $B[m]$ is a volume-constrained stable critical point of $\mathcal{F}_\alpha$ if and only if $\beta \leqslant \beta_*(\alpha')$, i.e.

$$m \leqslant |B_1|\beta_*\left(\frac{\alpha m^{1/3}}{|B_1|^{1/3}}\right).$$

Hence, $m_*(\alpha)$ is given implicitly by

$$m_*(\alpha) = |B_1|\beta_*\left(\frac{\alpha m_*(\alpha)^{1/3}}{|B_1|^{1/3}}\right). \tag{5.2}$$

Indeed, an explicit computation shows that $\beta_*(r)/r^3$ is decreasing, and the claim follows by taking $r = \alpha m^{1/3}/|B_1|^{1/3}$ and $r_* = \alpha m_*(\alpha)^{1/3}/|B_1|^{1/3}$.

Notice that although $\beta_*(\alpha)$ is explicit, we cannot explicitly find $m_*(\alpha)$. However, we can deduce some information about it. We now show that $m_*(\alpha) = +\infty$ for $\alpha \geqslant (2\pi)^{1/3}$. Rearranging (5.2) gives $\alpha = t/\beta_*(t)^{1/3}$ for $t = \alpha(m_*(\alpha)/|B_1|)^{1/3}$. Since $t/\beta_*(t)^{1/3}$ is increasing, it is injective and has a maximum value of $(2\pi)^{1/3}$ at $+\infty$. Hence $m_*(\alpha)$ is uniquely defined for $\alpha < (2\pi)^{1/3}$, and numerically computable via (5.2). Thus, if $\alpha \geqslant (2\pi)^{1/3}$ then $m_*(\alpha)$ is $+\infty$. Furthermore by implicit differentiation we see that

$$m_*'(\alpha) = \frac{|B_1|\beta_*'(t)\beta_*(t)}{\beta_*(t)^{2/3} - 1/3\, t\beta_*'(t)\beta_*(t)^{-1/3}} > 0,$$

and so $m_*(\alpha) \geqslant m_*(0) = 5$. □

Let us now turn to proving Theorem 5.1. We first expand the Yukawa potential in spherical harmonics. To this end let $\{Y_l^m\}$ be an orthonormal basis of ∂B_1 of spherical harmonics. Recall that if $x_1, x_2 \in \partial B_1$ then

$$\frac{e^{-\alpha|x_1-x_2|}}{|x_1-x_2|} = 4\pi\alpha \sum_{l=0}^{\infty} i_l(\alpha)k_l(\alpha) \sum_{m=-l}^{l} Y_l^m(\theta_1,\varphi_1)Y_l^{m^*}(\theta_2,\varphi_2) \tag{5.3}$$

where i_l and k_l are the modified spherical Bessel functions given by

$$i_l(\alpha) = \sqrt{\frac{\pi}{2\alpha}}\, I_{l+1/2}(\alpha) \quad \text{and} \quad k_l(\alpha) = \sqrt{\frac{2}{\pi\alpha}}\, K_{l+1/2}(\alpha)$$

and I and K are the modified Bessel functions of the first and second kind respectively (cf. [WDNWB21, Eq. (39)]). Let us define the operator

$$\mathscr{Y}_\alpha u(x) \coloneqq 2\int_{\partial B_1} (u(x)-u(y))Y_\alpha(x-y)\,\mathrm{d}\mathcal{H}^2(y)$$

so that for every $u \in C^1(\partial B_1)$ we have

$$[u]_\alpha^2 := \int_{\partial B_1}\int_{\partial B_1} |u(x)-u(y)|^2 Y_\alpha(x-y)\ \mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x) = \int_{\partial B_1} u(x)\,\mathscr{Y}_\alpha u(x)\,\mathrm{d}\mathcal{H}^2(x). \tag{5.4}$$

We aim to study the eigenvalues of $\mathscr{Y}_\alpha$. To this end we introduce the *modified Helmholtz operator on the sphere* as

$$\mathcal{Y}_\alpha u(x) := \int_{\partial B_1} u(y)Y_\alpha(x-y)\,\mathrm{d}\mathcal{H}^2(y)$$

and observe that

$$\mathscr{Y}_\alpha = 2(\mu_0^*(\alpha)\,\mathrm{Id} - \mathcal{Y}_\alpha)$$

where $\mu_0^*(\alpha)$ is the 0-th eigenvalue of $\mathcal{Y}_\alpha$ given by

$$\mu_0^*(\alpha) = \int_{\partial B_1} \frac{e^{-\alpha|x-y|}}{|x-y|}\,\mathrm{d}\mathcal{H}^2(y).$$

It suffices then to determine the eigenvalues $\mu_l^*(\alpha)$ of $\mathcal{Y}_\alpha$, that is for any Y_l^m with $m \in \{-l,\dots,l\}$ we have

$$\mathcal{Y}_\alpha Y_l^m = \mu_l^*(\alpha)Y_l^m.$$

Using (5.3) we find that

$$\mathcal{Y}_\alpha Y_l^m = \int_{\partial B_1} \frac{e^{-\alpha|x-y|}}{|x-y|} Y_l^m(y)\,\mathrm{d}\mathcal{H}^2(y) = 4\pi\alpha\, i_l(\alpha)k_l(\alpha)Y_l^m$$

and hence,

$$\mu_l^*(\alpha) = 4\pi\alpha\, i_l(\alpha)k_l(\alpha).$$

Consequently the eigenvalues μ_l^α of $\mathscr{Y}_\alpha$ are

$$\mu_l^\alpha = 2(\mu_0^*(\alpha) - \mu_l^*(\alpha)).$$

Let us collect some facts about the eigenvalues μ_l^α.

Lemma 5.2. *$\mu_0^\alpha = 0$, $(\mu_l^\alpha)_{l\in N}$ is increasing in l, and*

$$\mu_1^\alpha = \frac{4\pi e^{-\alpha}}{\alpha^3}\left(e^\alpha - (2\alpha^2+2\alpha+1)e^{-\alpha}\right), \tag{5.5}$$

$$\mu_2^\alpha = \frac{4\pi e^{-\alpha}}{\alpha^5}\left((3\alpha^2-9)e^\alpha + \left((\alpha^2+3\alpha+3)^2-\alpha^4\right)e^{-\alpha}\right). \tag{5.6}$$

In addition, we can write μ_1^α in terms of $\mathcal{N}_\alpha(B_1)$ and its derivatives as

$$\mu_1^\alpha = \frac{10\mathcal{N}_\alpha(B_1)+8\alpha\mathcal{N}_\alpha'(B_1)+\alpha^2\mathcal{N}_\alpha''(B_1)}{\mathcal{P}(B_1)}. \tag{5.7}$$

Proof. By definition

$$\mu_l^*(\alpha) = 4\pi\alpha i_l(\alpha)k_l(\alpha) = 4\pi I_{l+1/2}(\alpha)K_{l+1/2}(\alpha).$$

Next, [Seg11, Theorem 1] gives us that for any $\zeta \geqslant 0$ and $\alpha > 0$,

$$\frac{I_{\zeta+1/2}(\alpha)}{I_{\zeta-1/2}(\alpha)} < \frac{\alpha}{\zeta+\sqrt{\zeta^2+\alpha^2}} < \frac{K_{\zeta-1/2}(\alpha)}{K_{\zeta+1/2}(\alpha)},$$

that is, $I_{\zeta+1/2}(\alpha)K_{\zeta+1/2}(\alpha) < I_{\zeta-1/2}(\alpha)K_{\zeta-1/2}(\alpha)$. Taking $\zeta = l+1$ yields that $(\mu_l^*)_{l\in\mathbb{N}}$ is strictly decreasing; hence, $(\mu_l^\alpha)_{l\in N}$ is strictly increasing in l. Note that, by [AWH13, Eq. (14.196)] we have

$$\mu_0^*(\alpha) = \frac{2\pi e^{-\alpha}}{\alpha}\left(e^\alpha - e^{-\alpha}\right),$$
$$\mu_1^*(\alpha) = \frac{2\pi e^{-\alpha}}{\alpha^3}\left((\alpha^2-1)e^\alpha + (\alpha+1)^2e^{-\alpha}\right),$$
$$\mu_2^*(\alpha) = \frac{2\pi e^{-\alpha}}{\alpha^5}\left((\alpha^4-3\alpha^2+9)e^\alpha - (\alpha^2+3\alpha+3)^2e^{-\alpha}\right),$$

and the formulas (5.5) and (5.6) follow. As for (5.7) we can use Lemma 2.2, but there is an alternative proof without explicitly knowing $\mathcal{N}_\alpha(B_1)$ and its derivatives. As in [FFM+15, Proposition 8.4] we can appeal to the (almost) homogeneity of $\mathcal{N}_\alpha(B_r)$. On the one hand

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=1}\mathcal{N}_\alpha(B_r) &= \frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=1}\int_{B_1}\int_{B_1} r^5 Y_{\alpha r}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x\\ &= 5\mathcal{N}_\alpha(B_1) - \alpha\int_{B_1}\int_{B_1} e^{-\alpha|x-y|}\,\mathrm{d}x\,\mathrm{d}y = 5\mathcal{N}_\alpha(B_1)+\alpha\mathcal{N}_\alpha'(B_1).\end{aligned}$$

On the other hand, notice

$$\begin{aligned}\operatorname{div}(xY_\alpha(x)) &= \langle\nabla Y_\alpha(x), x\rangle + Y_\alpha(x)\operatorname{div}(x)\\ &= \left\langle \frac{\nabla e^{-\alpha|x|}}{|x|} - \frac{e^{-\alpha|x|}}{|x|^3}x, x\right\rangle + 3Y_\alpha(x) = \left\langle \nabla e^{-\alpha|x|}, \frac{x}{|x|}\right\rangle + 2Y_\alpha(x),\end{aligned}$$

so that by the divergence theorem,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=1}\mathcal{N}_\alpha(B_r) &= 2\int_{\partial B_1}\int_{B_1}\frac{e^{-\alpha|x-y|}}{|x-y|}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y)\\ &= \int_{\partial B_1}\int_{B_1}\operatorname{div}_x\left((x-y)Y_\alpha(x-y)\right) - \left\langle\nabla_x e^{-\alpha|x-y|}, \frac{x-y}{|x-y|}\right\rangle\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y)\end{aligned}$$

$$= \int_{\partial B_1} \int_{\partial B_1} \langle x-y, y\rangle Y_\alpha(x-y)\,\mathrm{d}\mathcal{H}^2(x) d\mathcal{H}^2(y) + \alpha \int_{B_1}\int_{\partial B_1} e^{-\alpha|x-y|}\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x$$

$$= \frac{1}{2}\int_{\partial B_1}\int_{\partial B_1} |x-y|e^{-\alpha|x-y|}\,\mathrm{d}x\,\mathrm{d}y + \alpha\int_{B_1}\int_{\partial B_1} e^{-\alpha|x-y|}\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x.$$

We recognize the first term as $1/2\mu_1^\alpha \mathcal{P}(B_1)$. Indeed, observe that the coordinate functions x_1, x_2, x_3 are harmonic on ∂B_1 and so for $i = 1, 2, 3$ we have $\mathscr{Y}_\alpha x_i = \mu_1^\alpha x_i$. Then, applying (5.4) to each of the x_i and summing the contributions gives for $x \in \partial B_1$ that

$$\mu_1^\alpha \mathcal{P}(B_1) = \sum_{i=1}^{3} \int_{\partial B_1} x_i \mathscr{Y}_\alpha x_i\,\mathrm{d}x = \sum_{i=1}^{3} [x_i]_\alpha^2 = [x]_\alpha^2 = \int_{\partial B_1}\int_{\partial B_1} |x-y|e^{-\alpha|x-y|}\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x). \quad (5.8)$$

Hence,

$$\mu_1^\alpha = \frac{10\mathcal{N}_\alpha(B_1) + 2\alpha\mathcal{N}'_\alpha(B_1)}{\mathcal{P}(B_1)} - \frac{2\alpha}{\mathcal{P}(B_1)}\int_{B_1}\int_{\partial B_1} e^{-\alpha|x-y|}\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x.$$

For the last term, observe that differentiating

$$2\int_{\partial B_1}\int_{B_1} \frac{e^{-\alpha|x-y|}}{|x-y|}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y) = \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=1} \mathcal{N}_\alpha(B_r) = 5\mathcal{N}_\alpha(B_1) + \alpha\mathcal{N}'_\alpha(B_1)$$

in α yields

$$-2\int_{\partial B_1}\int_{B_1} e^{-\alpha|x-y|}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(y) = 6\mathcal{N}'_\alpha(B_1) + \alpha\mathcal{N}''_\alpha(B_1),$$

and (5.7) follows. □

Now, we recall the second variation formulas of B_1 for the perimeter and the Yukawa potential obtained in [FFM+15].

Theorem 5.3 (cf. [FFM+15, Theorem 6.1])**.** *Let $X \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ be a volume preserving flow on B_1. Denoting $\zeta = \langle X, \nu_{B_1}\rangle$ then*

$$\delta^2\mathcal{P}(B_1)[X] = \int_{\partial B_1} |\nabla_\tau \zeta(z)|^2\,\mathrm{d}\mathcal{H}^2(z) - 2\int_{\partial B_1} \zeta(z)^2\,\mathrm{d}\mathcal{H}^2(z),$$

$$\delta^2\mathcal{N}_\alpha(B_1)[X] = -[\zeta]_\alpha^2 + \int_{\partial B_1} c^2_{Y_\alpha,\partial B_1}(z)\zeta(z)^2\,\mathrm{d}\mathcal{H}^2(z),$$

where

$$c^2_{Y_\alpha,\partial B_1}(x) := \int_{\partial B_1} Y_\alpha(x-y)|\nu_{B_1}(x) - \nu_{B_1}(y)|^2\,\mathrm{d}\mathcal{H}^2(y).$$

Let us determine $c^2_{Y_\alpha,\partial B_1}$ now. Observe first that it is a constant in x. Then since $\nu_{B_1}(x) = x$, by integrating over ∂B_1 and applying (5.8) we have

$$c^2_{Y_\alpha,\partial B_1}\mathcal{P}(B_1) = \int_{\partial B_1}\int_{\partial B_1} |x-y|e^{-\alpha|x-y|}\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x) = \mu_1^\alpha\mathcal{P}(B_1).$$

That is, $c^2_{Y_\alpha,\partial B_1} = \mu_1^\alpha$. Owing to this, we define the quadratic functionals

$$\mathcal{QP}(u) := \int_{\partial B_1} |\nabla_\tau u(z)|^2 \,\mathrm{d}\mathcal{H}^2(z) - \lambda_1 \int_{\partial B_1} u(z)^2 \,\mathrm{d}\mathcal{H}^2(z),$$
$$\mathcal{QN}_\alpha(u) := [u]_\alpha^2 - \mu_1^\alpha \int_{\partial B_1} u(z)^2 \,\mathrm{d}\mathcal{H}^2(z),$$

with $\lambda_l = l(l+1)$. These functionals are defined on the space

$$\tilde{H}^1(\partial B_1) := \left\{ u \in H^1(\partial B_1) \colon \int_{\partial B_1} u(x) \,\mathrm{d}\mathcal{H}^2(x) = 0 \right\}$$

which generalizes the space of normal components of volume preserving flows X on B_1. With this we have the following (analogous to [FFM+15, Proposition 7.2])

Proposition 5.4. *Let $\alpha \geqslant 0$ and $\beta > 0$. Then for every $u \in \tilde{H}^1(\partial B_1)$ it holds*

$$\mathcal{QP}(u) - \beta \mathcal{QN}_\alpha(u) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \Big((\lambda_l - \lambda_1) - \beta(\mu_l^\alpha - \mu_1^\alpha) \Big) |a_l^m(u)|^2, \tag{5.9}$$

where $a_l^m(u) = \int_{\partial B_1} u Y_l^{m*} \,\mathrm{d}\mathcal{H}^2$.

In particular, $\mathcal{QP}(u) - \beta \mathcal{QN}_\alpha(u) \geqslant 0$ for all $u \in \tilde{H}^1(\partial B_1)$ if and only if $\beta \in (0, \beta_*(\alpha)]$, where

$$\beta_*(\alpha) := \inf_{l \geqslant 2} \frac{\lambda_l - \lambda_1}{\mu_l^\alpha - \mu_1^\alpha}. \tag{5.10}$$

By Proposition A.1, the sequence of eigenvalue ratios is strictly increasing, and the infimum above is achieved when $l = 2$. Moreover, since $\lambda_l = l(l+1)$, by (5.5) and (5.6) we have

$$\beta_*(\alpha) = \frac{\lambda_2 - \lambda_1}{\mu_2^\alpha - \mu_1^\alpha} = \frac{\alpha^5 e^\alpha}{\pi \left[(2\alpha^2 - 9)e^\alpha + \left((\alpha^2 + 3\alpha + 3)^2 + \alpha^2(\alpha+1)^2 \right) e^{-\alpha} \right]}. \tag{5.11}$$

Now we are ready to prove the result that determines $\beta_*(\alpha)$ as the critical stability threshold.

Proof of Theorem 5.1. If $X \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ induces a volume-preserving flow, then by Theorem 5.3 the second variation of $\mathcal{E}_{\alpha,\beta}$ at B_1 with respect to X is

$$\delta^2 \mathcal{E}_{\alpha,\beta}(B_1)[X] = \mathcal{QP}(X \cdot \nu_{B_1}) - \beta \mathcal{QN}_\alpha(X \cdot \nu_{B_1}).$$

This means that B_1 is a volume-constrained stable set for $\mathcal{E}_{\alpha,\beta}$ if and only if

$$\mathcal{QP}(u) - \beta \mathcal{QN}_\alpha(u) \geqslant 0 \qquad \text{for every } u \in C^\infty(\partial B_1) \text{ with } \int_{\partial B_1} u \,\mathrm{d}\mathcal{H}^2 = 0. \tag{5.12}$$

Note that the condition (5.12) implies that $\delta^2 \mathcal{E}_{\alpha,\beta}(B_1)[X] \geqslant 0$ for every vector field X inducing a volume-preserving flow, since for every such vector field it must be $\int_{\partial B_1} X \cdot \nu_{B_1} \,\mathrm{d}\mathcal{H}^2 = 0$. The converse implication can be proved by arguing as in [FFM+15, Proof of Theorem 7.1]. So, the theorem follows by Proposition 5.4. $\square$

6. Rigidity of L^1-local minimizers

In this section we prove Theorem 1.5. As in Section 4, by Remark 1.6 it suffices to prove

Theorem 6.1. *Let $\alpha > 0$. If $\beta < \beta_*(\alpha)$ then B_1 is a local volume-constrained minimizer of $\mathcal{E}_{\alpha,\beta}$, that is there exists $\delta(\alpha,\beta) > 0$ such that if $|E\Delta B_1| \leqslant \delta$ and $|E| = |B_1|$ then*

$$\mathcal{E}_{\alpha,\beta}(B_1) \leqslant \mathcal{E}_{\alpha,\beta}(E)$$

and moreover if equality holds then $|E\Delta B_1| = 0$ up to translation. If instead $\beta > \beta_*(\alpha)$, then B_1 is not a local volume-constrained minimizer of $\mathcal{E}_{\alpha,\beta}$.

Proof of Theorem 1.5. For $m > 0$ and $\alpha \geqslant 0$ we have that

$$\mathcal{F}_\alpha(B) \leqslant \mathcal{F}_\alpha(F) \qquad \text{whenever} \quad |F| = m \text{ and } |F\Delta B[m]| \leqslant \varepsilon_* m$$

if and only if

$$\mathcal{E}_{\alpha',\beta}(B) \leqslant \mathcal{E}_{\alpha',\beta}(E) \qquad \text{whenever} \quad |E| = |B_1| \text{ and } |E\Delta B_1| \leqslant \varepsilon_*|B_1| \tag{6.1}$$

with $\alpha' = \alpha\beta^{1/3}$ and $\beta = m/|B_1|$. By Theorem 6.1, if $\beta < \beta_*(\alpha')$, i.e.

$$m < |B_1|\beta_*\left(\alpha(m/|B_1|)^{1/3}\right),$$

or equivalently $m < m_*(\alpha)$, then there exists $\delta(\alpha, m) > 0$ such that if $\varepsilon_* < \delta/|B_1|$ then (6.1) holds. □

Before we prove Theorem 6.1, we obtain a refinement of Lemma 4.2. To this end, for $u \in H^1(\partial B_1)$, recall that $[u]^2_\alpha$ is defined in (5.4) by

$$[u]^2_\alpha = \int_{\partial B_1}\int_{\partial B_1} |u(x) - u(y)|^2 Y_\alpha(x-y)\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x).$$

Then for nearly spherical sets we have the following control on the nonlocal term of $\mathcal{E}_{\alpha,\beta}$.

Lemma 6.2. *There exist $C > 0$ and $t_0 > 0$, independent of α, so that if $E \subset \mathbb{R}^3$ is an open set with $|E| = |B_1|$ and*

$$\partial E = \left\{(1 + t\,u(x))x \,:\, x \in \partial B_1\right\}$$

for some $u \in C^1(\partial B_1)$ with $\|u\|_{C^1(\partial B_1)} \leqslant 1/2$ and $t \in (0, 2t_0)$, then

$$\mathcal{N}_\alpha(B_1) - \mathcal{N}_\alpha(E) \leqslant \left([u]^2_\alpha - \mu_1^\alpha\|u\|^2_{L^2(\partial B_1)}\right)\frac{t^2}{2} + \left(\frac{C}{2}[u]^2_\alpha + B(\alpha)\|u\|^2_{L^2(\partial B_1)}\right)t^3$$

where $B(\alpha)$ is a positive function defined by

$$B(\alpha) := \max\left\{B_1(\alpha) + B_2(\alpha), B_2(\alpha)\right\}, \quad B_1(\alpha) := \frac{15\alpha^2\mathcal{N}''_\alpha(B_1) + 5\alpha^3\mathcal{N}'''_\alpha(B_1)}{24\mathcal{P}(B_1)},$$

$$B_2(\alpha) := -\frac{120\alpha\mathcal{N}'_\alpha(B_1) + 90\alpha^2\mathcal{N}''_\alpha(B_1) + 25\alpha^3\mathcal{N}'''_\alpha(B_1)}{48\mathcal{P}(B_1)}.$$

Proof. Introduce, for $r, \rho, \theta \geqslant 0$, the function

$$f^\alpha_\theta(r,\rho) = \frac{r^2\rho^2 e^{-\alpha(|r-\rho|^2 + r\rho\theta^2)^{1/2}}}{(|r-\rho|^2 + r\rho\theta^2)^{1/2}}.$$

With this notation we have

$$\mathcal{N}_\alpha(E) = \int_{\partial B_1}\int_{\partial B_1}\int_0^{1+tu(x)}\int_0^{1+tu(y)} f^\alpha_{|x-y|}(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x).$$

We can rearrange the integrals to instead write

$$\begin{aligned}\mathcal{N}_\alpha(E) = &\int_{\partial B_1}\int_{\partial B_1}\int_0^{1+tu(x)}\int_0^{1+tu(x)} f^\alpha_{|x-y|}(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x)\\ &-\frac{1}{2}\int_{\partial B_1}\int_{\partial B_1}\int_{1+tu(y)}^{1+tu(x)}\int_{1+tu(y)}^{1+tu(x)} f^\alpha_{|x-y|}(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x).\end{aligned} \tag{6.2}$$

We start by estimating the first term. Observe that unlike in [FFM+15, Lemma 5.3] the function f^α_θ is not homogeneous. In particular, we instead have for $\lambda > 0$ that

$$f^\alpha_\theta(\lambda r,\lambda\rho) = \lambda^3 f^{\alpha\lambda}_\theta(r,\rho).$$

Consequently, after a change of variables, we have

$$\begin{aligned}&\int_{\partial B_1}\int_0^{1+tu(x)}\int_0^{1+tu(x)} f^\alpha_{|x-y|}(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^2(y)\\ &\qquad = (1+tu(x))^5\int_{\partial B_1}\int_0^1\int_0^1 f^{\alpha(1+tu(x))}_{|x-y|}(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^2(y).\end{aligned} \tag{6.3}$$

Using the inequality $e^{-s} \geqslant 1-s+s^2/2-s^3/6$ with $s = \alpha(|r-\rho|^2+r\rho|x-y|^2)^{1/2}u(x)t$ we get

$$\begin{aligned}f^{\alpha(1+tu(x))}_{|x-y|}(r,\rho) \geqslant\ & f^\alpha_{|x-y|}(r,\rho) + \left(\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}f^\alpha_{|x-y|}(r,\rho)\right)u(x)\,t\\ &+\left(\frac{\alpha^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}f^\alpha_{|x-y|}(r,\rho)\right)u(x)^2\,t^2 + \left(\frac{\alpha^3}{6}\frac{\mathrm{d}^3}{\mathrm{d}\alpha^3}f^\alpha_{|x-y|}(r,\rho)\right)u(x)^3\,t^3.\end{aligned}$$

We integrate this to get

$$\begin{aligned}&\int_{\partial B_1}\int_0^1\int_0^1 f^{\alpha(1+tu(x))}_{|x-y|}(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^2(y)\\ &\qquad\geqslant \frac{\mathcal{N}_\alpha(B_1)}{\mathcal{P}(B_1)} + \frac{\alpha\mathcal{N}'_\alpha(B_1)}{\mathcal{P}(B_1)}u(x)\,t + \frac{\alpha^2\mathcal{N}''_\alpha(B_1)}{2\mathcal{P}(B_1)}u(x)^2\,t^2 + \frac{\alpha^3\mathcal{N}'''_\alpha(B_1)}{6\mathcal{P}(B_1)}u(x)^3\,t^3.\end{aligned} \tag{6.4}$$

Indeed, we notice that the following integral is independent of x, and so

$$\int_{\partial B_1}\int_0^1\int_0^1 f^\alpha_{|x-y|}(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^2(y) = \frac{\mathcal{N}_\alpha(B_1)}{\mathcal{P}(B_1)}.$$

Thus combining (6.3) and (6.4) we get

$$\begin{aligned}&\int_{\partial B_1}\int_{\partial B_1}\int_0^{1+tu(x)}\int_0^{1+tu(x)} f^\alpha_{|x-y|}(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x)\\ &\qquad\geqslant \int_{\partial B_1}(1+tu(x))^5\sum_{k=0}^{3} I_k(\alpha)u(x)^k t^k\,\mathrm{d}\mathcal{H}^2(x)\end{aligned} \tag{6.5}$$

where we define $I_k(\alpha)$ by

$$I_k(\alpha) = \frac{\alpha^k\mathcal{N}^{(k)}_\alpha(B_1)}{k!\,\mathcal{P}(B_1)}.$$

By the volume constraint $|E| = |B_1|$ we have

$$\int_{\partial B_1} (1+tu(x))^3 \,\mathrm{d}\mathcal{H}^2(x) = \mathcal{P}(B_1). \tag{6.6}$$

Applying this and (6.5) in (6.2) then yields

$$\begin{aligned} \mathcal{N}_\alpha(B_1) - \mathcal{N}_\alpha(E) \leqslant \frac{t^2}{2} g(t) - \int_{\partial B_1} (1+tu(x))^3 \sum_{k=1}^{3} I_k(\alpha) u(x)^k t^k \,\mathrm{d}\mathcal{H}^2(x) \\ + \int_{\partial B_1} (1+tu(x))^3 (1-(1+tu(x))^2) \sum_{k=0}^{3} I_k(\alpha) u(x)^k t^k \,\mathrm{d}\mathcal{H}^2(x), \end{aligned} \tag{6.7}$$

where

$$g(t) = \int_{\partial B_1} \int_{\partial B_1} \int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} f^{\alpha}_{|x-y|}(1+tr, 1+t\rho) \,\mathrm{d}\rho \,\mathrm{d}r \,\mathrm{d}\mathcal{H}^2(y) \,\mathrm{d}\mathcal{H}^2(x).$$

To estimate the first term in (6.7), observe that

$$g(0) = \int_{\partial B_1} \int_{\partial B_1} |u(x) - u(y)|^2 Y_\alpha(x-y) \,\mathrm{d}\mathcal{H}^2(y) \,\mathrm{d}\mathcal{H}^2(x) = [u]_\alpha^2.$$

Moreover, arguing as in [FFM+15, Appendix D], we find a constant $C > 0$, independent of α, such that for $\tau \in (0,t)$ we have $g(t) = g(0) + g'(\tau)t$ and $|g'(\tau)| \leqslant C g(0)$. To estimate the second term in (6.7), we first expand $(1+tu(x))^3$ to get

$$\begin{aligned} &- \int_{\partial B_1} (1+tu(x))^3 \sum_{k=1}^{3} I_k(\alpha) u(x)^k t^k \,\mathrm{d}\mathcal{H}^2(x) \\ &\quad = -I_1(\alpha) [\![u]\!]_1 \, t - (3I_1(\alpha) + I_2(\alpha)) \, [\![u]\!]_2 \, t^2 - (3I_1(\alpha) + 3I_2(\alpha) + I_3(\alpha)) \, [\![u]\!]_3 \, t^3 \\ &\qquad - (I_1(\alpha) + 3I_2(\alpha) + 3I_3(\alpha)) \, [\![u]\!]_4 \, t^4 - (I_2(\alpha) + 3I_3(\alpha)) \, [\![u]\!]_5 \, t^5 - I_3(\alpha) [\![u]\!]_6 \, t^6, \end{aligned} \tag{6.8}$$

where $[\![u]\!]_k \coloneqq \int_{\partial B_1} u(x)^k \,\mathrm{d}\mathcal{H}^2(x)$. Next, notice that (6.6) further implies

$$-t[\![u]\!]_1 = t^2 [\![u]\!]_2 + \frac{t^3}{3} [\![u]\!]_3, \tag{6.9}$$

and applying this in (6.8) yields

$$\begin{aligned} &- \int_{\partial B_1} (1+tu(x))^3 \sum_{k=1}^{3} I_k(\alpha) u(x)^k t^k \,\mathrm{d}\mathcal{H}^2(x) \\ &\quad = -(2I_1(\alpha) + I_2(\alpha)) \, [\![u]\!]_2 \, t^2 - \left(\frac{8}{3} I_1(\alpha) + 3I_2(\alpha) + I_3(\alpha) \right) [\![u]\!]_3 \, t^3 \\ &\qquad - (I_1(\alpha) + 3I_2(\alpha) + 3I_3(\alpha)) \, [\![u]\!]_4 \, t^4 - (I_2(\alpha) + 3I_3(\alpha)) \, [\![u]\!]_5 \, t^5 - I_3(\alpha) [\![u]\!]_6 \, t^6. \end{aligned} \tag{6.10}$$

For the third term in (6.7), if t_0 is as in Lemma 2.2(ii) then we have for $s \in (-t_0, t_0)$ that

$$\mathcal{N}_\alpha(B_1) + \alpha \mathcal{N}'_\alpha(B_1) s + \alpha^3 \mathcal{N}'''_\alpha(B_1) \frac{s^3}{6} \geqslant 0.$$

As $\|u\|_{L^\infty(\partial B_1)} \leqslant 1/2$ and $t < 2t_0$, this holds in particular for $s = u(x)t$. Since $\mathcal{N}''_\alpha(B_1) > 0$ by Lemma 2.2(iii), applying the above gives $\sum_{k=0}^3 I_k(\alpha)u(x)^k t^k > 0$. Thus, we can estimate

$$\int_{\partial B_1} (1+tu(x))^3\big(1-(1+tu(x))^2\big)\sum_{k=0}^{3} I_k(\alpha)u(x)^k t^k \,\mathrm{d}\mathcal{H}^2(x) \leqslant \int_{\partial B_1}\big(-2tu(x)-7t^2u(x)^2-9t^3u(x)^3\big)\sum_{k=0}^{3} I_k(\alpha)u(x)^k t^k \,\mathrm{d}\mathcal{H}^2(x).$$

Expanding the right-hand side then yields

$$\begin{aligned}\int_{\partial B_1} &(1+tu(x))^3\big(1-(1+tu(x))^2\big)\sum_{k=0}^{3} I_k(\alpha)u(x)^k t^k \,\mathrm{d}\mathcal{H}^2(x)\\ &\leqslant -2I_0(\alpha)[\![u]\!]_1\, t - (7I_0(\alpha)+2I_1(\alpha))\,[\![u]\!]_2\, t^2 - (9I_0(\alpha)+7I_1(\alpha)+2I_2(\alpha))\,[\![u]\!]_3\, t^3\\ &\quad - (9I_1(\alpha)+7I_2(\alpha)+2I_3(\alpha))\,[\![u]\!]_4\, t^4 - (9I_2(\alpha)+7I_3(\alpha))\,[\![u]\!]_5\, t^5 - 9I_3(\alpha)[\![u]\!]_6\, t^6.\end{aligned}$$

We again apply (6.9) to obtain

$$\begin{aligned}\int_{\partial B_1} &(1+tu(x))^3\big(1-(1+tu(x))^2\big)\sum_{k=0}^{3} I_k(\alpha)u(x)^k t^k \,\mathrm{d}\mathcal{H}^2(x)\\ &\leqslant -(5I_0(\alpha)+2I_1(\alpha))\,[\![u]\!]_2\, t^2 - \left(\frac{25}{3}I_0(\alpha)+7I_1(\alpha)+2I_2(\alpha)\right)[\![u]\!]_3\, t^3\\ &\quad - (9I_1(\alpha)+7I_2(\alpha)+2I_3(\alpha))\,[\![u]\!]_4\, t^4\\ &\quad - (9I_2(\alpha)+7I_3(\alpha))\,[\![u]\!]_5\, t^5 - 9I_3(\alpha)[\![u]\!]_6\, t^6.\end{aligned} \tag{6.11}$$

Combining (6.7), $g(t) = g(0) + Cg(0)t$, (6.10), (6.11), and $[\![u]\!]_k \leqslant 1/2^{k-2}\|u\|^2_{L^2(\partial(B_1))}$ for $k \geqslant 3$, we finally have

$$\begin{aligned}\mathcal{N}_\alpha(B_1) - \mathcal{N}_\alpha(E) &\leqslant \Big([u]_\alpha^2 - (10I_0(\alpha)+8I_1(\alpha)+2I_2(\alpha))\,\|u\|^2_{L^2(\partial B_1)}\Big)\frac{t^2}{2}\\ &\quad + \left(\frac{C}{2}[u]_\alpha^2 + \frac{1}{2}\left(\frac{25}{3}I_0(\alpha)+\frac{29}{3}I_1(\alpha)+5I_2(\alpha)+I_3(\alpha)\right)^- \|u\|^2_{L^2(\partial B_1)}\right)t^3\\ &\quad + \frac{1}{4}\,(10I_1(\alpha)+10I_2(\alpha)+5I_3(\alpha))^-\,\|u\|^2_{L^2(\partial B_1)}\, t^4\\ &\quad + \frac{1}{8}\,(10I_2(\alpha)+10I_3(\alpha))^-\,\|u\|^2_{L^2(\partial B_1)}\, t^5 + \frac{1}{16}\,(10I_3(\alpha))^-\,\|u\|^2_{L^2(\partial B_1)}\, t^6.\end{aligned}$$

Now, by Lemma 2.2(iii),

$$\frac{25}{3}I_0(\alpha)+\frac{29}{3}I_1(\alpha)+5I_2(\alpha)+I_3(\alpha) = \frac{50\mathcal{N}_\alpha(B_1)+58\alpha\mathcal{N}'_\alpha(B_1)+15\alpha^2\mathcal{N}''_\alpha(B_1)+\alpha^3\mathcal{N}'''_\alpha(B_1)}{6\mathcal{P}(B_1)} > 0,$$

and

$$10I_1(\alpha)+10I_2(\alpha)+5I_3(\alpha) = \frac{60\alpha\mathcal{N}'_\alpha(B_1)+30\alpha^2\mathcal{N}''_\alpha(B_1)+5\alpha^3\mathcal{N}'''_\alpha(B_1)}{6\mathcal{P}(B_1)} < 0,$$

$$10 I_3(\alpha) = \frac{5\alpha^3 \mathcal{N}_\alpha'''(B_1)}{3\mathcal{P}(B_1)} < 0.$$

So we have

$$\begin{aligned}\mathcal{N}_\alpha(B_1) - \mathcal{N}_\alpha(E) \leqslant \Big([u]_\alpha^2 - (10I_0(\alpha) + 8I_1(\alpha) + 2I_2(\alpha))\, \|u\|^2_{L^2(\partial B_1)}\Big)\frac{t^2}{2} + \frac{C}{2}[u]_\alpha^2\, t^3 \\ - \frac{10I_1(\alpha) + 10I_2(\alpha) + 5I_3(\alpha)}{4}\|u\|^2_{L^2(\partial B_1)}\, t^4 \\ + \frac{(5I_2(\alpha) + 5I_3(\alpha))^-}{4}\|u\|^2_{L^2(\partial B_1)}\, t^5 - \frac{5I_3(\alpha)}{8}\|u\|^2_{L^2(\partial B_1)}\, t^6.\end{aligned}$$

Up to decreasing t_0, we may assume $t < 1$; hence,

$$\begin{aligned}\mathcal{N}_\alpha(B_1) - \mathcal{N}_\alpha(E) \leqslant \Big([u]_\alpha^2 - (10I_0(\alpha) + 8I_1(\alpha) + 2I_2(\alpha))\, \|u\|^2_{L^2(\partial B_1)}\Big)\frac{t^2}{2} \\ + \left(\frac{C}{2}[u]_\alpha^2 - \frac{1}{8}\left(20I_1(\alpha) + 20I_2(\alpha) + 15I_3(\alpha) - (10I_2(\alpha) + 10I_3(\alpha))^-\right)\|u\|^2_{L^2(\partial B_1)}\right) t^3.\end{aligned}$$

This implies the result since by (5.7) we have

$$10I_0(\alpha) + 8I_1(\alpha) + 2I_2(\alpha) = \frac{10\mathcal{N}_\alpha(B_1) + 8\alpha\mathcal{N}_\alpha'(B_1) + \alpha^2\mathcal{N}_\alpha''(B_1)}{\mathcal{P}(B_1)} = \mu_1^\alpha,$$

and by direct computation

$$\begin{aligned}-\frac{1}{8}\left(20I_1(\alpha) + 20I_2(\alpha) + 15I_3(\alpha) - (10I_2(\alpha) + 10I_3(\alpha))^-\right) \\ = \frac{1}{8}\max\{-20I_1(\alpha) + 30I_2(\alpha) + 25I_3(\alpha), -20I_1(\alpha) + 20I_2(\alpha) + 15I_3(\alpha)\} = B(\alpha),\end{aligned}$$

as desired. □

Next we prove the following theorem which provides a quantitative bound on the energy deficit.

Theorem 6.3. *There exists $C > 0$, independent of α, such that for every $\alpha \geqslant 0$ and $\beta \in (0, \beta_*(\alpha))$ there exists $\varepsilon_\beta(\alpha) > 0$ such that if E is a nearly spherical set with $|E| = |B_1|$, $\int_E x\,\mathrm{d}x = 0$, and $\|u\|_{C^1(\partial B_1)} < \varepsilon_\beta$, then*

$$\mathcal{E}_{\alpha,\beta}(E) - \mathcal{E}_{\alpha,\beta}(B_1) \geqslant C\left(1 - \frac{\beta}{\beta_*(\alpha)}\right)\|u\|^2_{H^1(\partial B_1)}.$$

Moreover, $\varepsilon_\beta(\alpha)$ can be chosen of the form

$$\varepsilon_\beta(\alpha) = \frac{\varepsilon_0}{\ell(\alpha)}\left(1 - \frac{\beta}{\beta_*(\alpha)}\right)$$

for some $\varepsilon_0 > 0$ and explicitly computable linear function $\ell(\alpha)$ with $\ell(0) > 0$ and $\ell'(0) > 0$. In particular, $\varepsilon_\beta(\alpha)$ does not degenerate as $\alpha \to 0^+$.

Proof. As in Lemma 6.2 we consider $u \in C^1(\partial B_1)$ with $\|u\|_{C^1(\partial B_1)} \leqslant 1/2$ and $t \in (0, 2\varepsilon_\beta)$ so that the set $E \subset \mathbb{R}^3$ with boundary

$$\partial E = \Big\{(1 + tu(x))x \colon x \in \partial B_1\Big\}$$

satisfies $|E| = |B_1|$ and $\int_E x \,\mathrm{d}x = 0$. If ε_β is small enough, then for some $C > 0$

$$\mathcal{E}_{\alpha,\beta}(E) - \mathcal{E}_{\alpha,\beta}(B_1) \geqslant \frac{t^2}{2} \left(\mathcal{QP}(u) - \beta \mathcal{QN}_\alpha(u) \right) - Ct^3 \left(\int_{\partial B_1} |\nabla_\tau u|^2 \,\mathrm{d}\mathcal{H}^2 + \lambda_1 \|u\|^2_{L^2} + \beta \left([u]^2_\alpha + \frac{B(\alpha)}{C} \right) \|u\|^2_{L^2(\partial B_1)} \right).$$

Recall now from [FFM+15, Eqs. (2.25) and (2.28)] that for ε_β small enough we have

$$\frac{1}{2} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} |a_l^m(u)|^2 - \sum_{m=-1}^{1} |a_1^m(u)|^2 - |a_0(u)|^2 \geqslant \frac{1}{4} \|u\|^2_{L^2(\partial B_1)}, \tag{6.12}$$

$$\int_{\partial B_1} |\nabla_\tau u|^2 \,\mathrm{d}\mathcal{H}^2 - \lambda_1 \|u\|^2_{L^2(\partial B_1)} \geqslant \frac{1}{4} \int_{\partial B_1} |\nabla_\tau u|^2 \,\mathrm{d}\mathcal{H}^2 + \frac{\lambda_1}{4} \|u\|^2_{L^2(\partial B_1)}. \tag{6.13}$$

We estimate first $\mathcal{QP}(u) - \beta \mathcal{QN}_\alpha(u)$. By (5.9) and the definition of $\beta_*(\alpha)$ in (5.10)

$$\begin{aligned} \mathcal{QP}(u) - \beta \mathcal{QN}_\alpha(u) &= \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left((\lambda_l - \lambda_1) - \beta(\mu_l^\alpha - \mu_1^\alpha) \right) |a_l^m(u)|^2 \\ &\geqslant \left(1 - \frac{\beta}{\beta_*(\alpha)} \right) \sum_{l=2}^{\infty} \sum_{m=-l}^{l} (\lambda_l - \lambda_1) |a_l^m(u)|^2. \end{aligned}$$

Since

$$\sum_{l=2}^{\infty} \sum_{m=-l}^{l} (\lambda_l - \lambda_1) |a_l^m(u)|^2 = \int_{\partial B_1} |\nabla_\tau u|^2 \,\mathrm{d}\mathcal{H}^2 - \lambda_1 \|u\|^2_{L^2(\partial B_1)},$$

by applying (6.13) we therefore have

$$\mathcal{QP}(u) - \beta \mathcal{QN}_\alpha(u) \geqslant \frac{1}{4} \left(1 - \frac{\beta}{\beta_*(\alpha)} \right) \left(\int_{\partial B_1} |\nabla_\tau u|^2 \,\mathrm{d}\mathcal{H}^2 + \lambda_1 \|u\|^2_{L^2(\partial B_1)} \right). \tag{6.14}$$

In particular this implies $\mathcal{QP}(u) \geqslant \beta \mathcal{QN}_\alpha(u)$. We now control the cubic terms. From Lemma 5.2 we have

$$\frac{\mu_2^\alpha}{\mu_1^\alpha} = \frac{(3\alpha^2 - 9)e^\alpha + \left((\alpha^2 + 3\alpha + 3)^2 - \alpha^4 \right) e^{-\alpha}}{\alpha^2 e^\alpha - (2\alpha^4 + 2\alpha^3 + \alpha^2) e^{-\alpha}} \geqslant \frac{6}{5}$$

This follows since the above is equivalent to

$$(3\alpha^2 - 15) e^{2\alpha} \geqslant -15 - 30\alpha - 27\alpha^2 - 14\alpha^3 - 4\alpha^4,$$

and noting that the right-hand side is the fourth order Taylor expansion of the left-hand side with a positive remainder. Then, observe as in [FFM+15, Eq. (8.9)] that, from (6.12), the above, and Lemma 5.2, we have

$$\mu_1^\alpha \|u\|_{L^2(\partial B_1)} \leqslant 2\mu_1^\alpha \sum_{l=2}^{\infty} \sum_{m=-l}^{l} |a_l^m(u)|^2 \leqslant 10 \sum_{l=2}^{\infty} \sum_{m=-l}^{l} (\mu_l^\alpha - \mu_1^\alpha) |a_l^m(u)|^2 = 10 \mathcal{QN}_\alpha(u).$$

We are thus reduced to controlling $B(\alpha)$ in terms of μ_1^α. From Lemma 2.2(i) and (5.5), we deduce

$$B(\alpha) \leqslant \frac{25}{12} (\alpha + 1) \mu_1^\alpha,$$

since in particular for α sufficiently large $B(\alpha) = B_2(\alpha)$. Hence, for some $c_1, c_2 > 0$

$$\begin{aligned}\beta\left([u]_\alpha^2 + \frac{B(\alpha)}{C}\|u\|^2_{L^2(\partial B_1)}\right) &= \beta\mathcal{QN}_\alpha(u) + \beta\left(\mu_1^\alpha + \frac{B(\alpha)}{C}\right)\|u\|^2_{L^2(\partial B_1)} \\ &\leqslant (c_1\alpha + c_2)\,\beta\mathcal{QN}_\alpha(u) \leqslant (c_1\alpha + c_2)\,\mathcal{QP}(u). \qquad (6.15)\end{aligned}$$

Again applying (6.13) we have

$$\mathcal{QP}(u) = \int_{\partial B_1} |\nabla_\tau u|^2 \,\mathrm{d}\mathcal{H}^2 - \lambda_1\|u\|^2_{L^2(\partial B_1)} \geqslant \frac{1}{4}\left(\int_{\partial B_1} |\nabla_\tau u|^2 \,\mathrm{d}\mathcal{H}^2 + \lambda_1\|u\|^2_{L^2(\partial B_1)}\right).$$

Combining this with (6.14) and (6.15) we have thus shown

$$\mathcal{E}_{\alpha,\beta}(E) - \mathcal{E}_{\alpha,\beta}(B_1) \geqslant \left(\frac{t^2}{8}\left(1 - \frac{\beta}{\beta_*(\alpha)}\right) - \ell(\alpha)t^3\right)\left(\int_{\partial B_1} |\nabla_\tau u|^2 \,\mathrm{d}\mathcal{H}^2 + \lambda_1\|u\|^2_{L^2(\partial B_1)}\right)$$

for some linear function $\ell(\alpha)$ with $\ell(0), \ell'(0) > 0$. Therefore, for sufficiently small t depending on α, we have

$$\mathcal{E}_{\alpha,\beta}(E) - \mathcal{E}_{\alpha,\beta}(B_1) \geqslant ct^2\left(1 - \frac{\beta}{\beta_*(\alpha)}\right)\|u\|^2_{H^1(\partial B_1)},$$

as desired. □

We now prove Theorem 6.1. The core of the proof is essentially the same as in [FFM+15, Lemma 8.5], but with some new ideas to overcome the lack of homogeneity in $\mathcal{N}_\alpha$.

Proof of Theorem 6.1. Since minimality implies stability, if B_1 is a local volume-constrained minimizer of $\mathcal{E}_{\alpha,\beta}$ then $\beta \leqslant \beta_*(\alpha)$. So we only need to prove that if $\beta < \beta_*(\alpha)$ then B_1 is a local volume-constrained minimizer of $\mathcal{E}_{\alpha,\beta}$. Supposing not, then there exists $\beta < \beta_*(\alpha)$ and a sequence $\{E_i\}_{i=1}^\infty$ of sets of finite perimeter with

$$|E_i| = |B_1| \quad \text{and} \quad |E_i \Delta B_1| \to 0, \quad \text{but} \quad \mathcal{E}_{\alpha,\beta}(E_i) \leqslant \mathcal{E}_{\alpha,\beta}(B_1). \qquad (6.16)$$

The idea now is to use a selection principle type argument to generate a sequence F_i with better regularity that allows us to apply Theorem 6.3 to derive a contradiction.

Step one: We first show that the sequence $\{E_i\}_{i=1}^\infty$ above can be chosen such that $E_i \subset B_R$ for some $R > 0$. Fix $\eta > 0$, to be chosen later, so that for i large enough we have $|E_i \Delta B_1| < \eta$. By the truncation lemma [AJ76, VI. 14], see also [Mag12, Lemma 29.12], there exists $c_1, c_2 > 0$ and $r_i > 0$ such that $1 \leqslant r_i \leqslant 1 + c_1\eta^{1/3}$ and

$$\mathcal{P}(E_i \cap B_{r_i}) \leqslant \mathcal{P}(E_i) - \frac{|E_i \backslash B_{r_i}|}{c_2\eta^{1/3}}. \qquad (6.17)$$

Now let λ_i be such that $E_i' := \lambda_i(E_i \cap B_{r_i})$ satisfies $|E_i'| = |B_1|$. We aim to show $\{E_i'\}_{i=1}^\infty$ still satisfies (6.16), while being uniformly bounded in some ball. Since $r_i \to 1^+$ and $|E_i \Delta B_1| \to 0$, it follows that $\lambda_i \to 1^+$ and $|E_i' \Delta B_1| \to 0$. In particular, for i large enough we have that $E_i' \subset B_R$ for $R = 2 + c_1\eta^{1/3}$. We now show $\mathcal{E}_{\alpha,\beta}(E_i') \leqslant \mathcal{E}_{\alpha,\beta}(B_1)$. Since $\lambda_i \geqslant 1$, we have $\mathcal{N}_{\alpha\lambda_i}(E) \leqslant \mathcal{N}_\alpha(E)$. Hence for λ_i sufficiently close to 1 we have

$$\begin{aligned}\mathcal{E}_{\alpha,\beta}(E_i') &= \lambda_i^5\left(\lambda_i^{-3}\mathcal{P}(E_i \cap B_{r_i}) + \beta\mathcal{N}_{\alpha\lambda_i}(E_i \cap B_{r_i})\right) \\ &\leqslant (1 + C|E_i \backslash B_{r_i}|)\left(\mathcal{P}(E_i \cap B_{r_i}) + \beta\mathcal{N}_\alpha(E_i \cap B_{r_i})\right).\end{aligned}$$

Next, applying (6.17), $\mathcal{N}_\alpha(E_i \cap B_{r_i}) \leqslant \mathcal{N}_\alpha(E_i)$, and (6.16) we further estimate

$$\mathcal{E}_{\alpha,\beta}(E_i') \leqslant \mathcal{E}_{\alpha,\beta}(E_i) + C\left(\mathcal{E}_{\alpha,\beta}(B_1) - \frac{1}{c_2\,\eta^{1/3}}\right)|E_i \backslash B_{r_i}|.$$

Hence if η was chosen so that $\eta^{1/3} < 1/(c_2\,\mathcal{E}_{\alpha,\beta}(B_1))$ then the term in parentheses is negative and we conclude $\mathcal{E}_{\alpha,\beta}(E_i') \leqslant \mathcal{E}_{\alpha,\beta}(E_i) \leqslant \mathcal{E}_{\alpha,\beta}(B_1)$.

Step two: Assuming now that the sequence $\{E_i\}_{i=1}^\infty$ additionally satisfies $E_i \subset B_R$ with $R > 0$ as above, we aim to generate a new sequence $\{F_i\}_{i=1}^\infty$ with better regularity, namely that the F_i are nearly spherical. To this end for $\Lambda > 0$ we consider the variational problem for each $i \in \mathbb{N}$

$$\inf\left\{\mathcal{E}_{\alpha,\beta}(E) + \Lambda\,|E\Delta E_i| \mid E \subset \mathbb{R}^3\right\}. \tag{6.18}$$

Note that in the above we assume no volume constraint nor that a priori $E \subset B_R$. We first show these problems admit a minimizer. Indeed, if $F = E \cap B_R$ then $\mathcal{E}_{\alpha,\beta}(F) \leqslant \mathcal{E}_{\alpha,\beta}(E)$ by trivial inclusions and $|F\Delta E_i| \leqslant |E\Delta E_i|$. Hence we can assume $E \subset B_R$, and a minimizer F_i exists with $F_i \subset B_R$. We now claim that there exists a $\Lambda' > 0$ such that

$$\mathcal{P}(F_i) \leqslant \mathcal{P}(E) + \Lambda'\,|E\Delta F_i| \qquad \text{for all } E \subset \mathbb{R}^3. \tag{6.19}$$

First if $E \subset \mathbb{R}^3$ then

$$\mathcal{N}_\alpha(E) - \mathcal{N}_\alpha(F_i) \leqslant 2\int_E \int_{E\backslash F_i} \frac{e^{-\alpha|x-y|}}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \leqslant 2|E\backslash F_i| \int_{B_r} \frac{e^{-\alpha|z|}}{|z|}\,\mathrm{d}z \tag{6.20}$$

where $r = (|E|/|B_1|)^{1/3}$. The latter integral is estimated as

$$\int_{B_r} \frac{e^{-\alpha|z|}}{|z|}\,\mathrm{d}z = \frac{4\pi(1-(1+\alpha r)e^{-\alpha r})}{\alpha^2} \leqslant \frac{4\pi}{\alpha^2}.$$

Hence, by (6.18) we get

$$\mathcal{P}(F_i) \leqslant \mathcal{P}(E) + \beta\mathcal{N}_\alpha(E) - \beta\mathcal{N}_\alpha(F_i) + \Lambda\,(|E\Delta E_i| - |F_i\Delta E_i|) \leqslant \mathcal{P}(E) + \left(\frac{8\pi\beta}{\alpha^2} + \Lambda\right)|E\Delta F_i|$$

which proves (6.19) for $\Lambda' = 8\pi\beta/\alpha^2 + \Lambda$. Next, by testing (6.18) with E_i and applying (6.16) we get

$$\mathcal{E}_{\alpha,\beta}(F_i) + \Lambda\,|F_i\Delta E_i| \leqslant \mathcal{E}_{\alpha,\beta}(E_i) \leqslant \mathcal{E}_{\alpha,\beta}(B_1) \tag{6.21}$$

which implies by the triangle inequality, since $|E_i\Delta B_1| \to 0$, that for sufficiently large i

$$|F_i\Delta B_1| \leqslant \frac{2\mathcal{E}_{\alpha,\beta}(B_1)}{\Lambda}.$$

By standard regularity theory, see for instance [Mag12, Theorem 26.6], by choosing Λ (depending on $\alpha, \beta > 0$) sufficiently large there exists $u_i \in C^{1,\gamma}(\partial B_1)$ where

$$\partial F_i = \Big\{(1+u_i(x))x \,:\, x \in \partial B_1\Big\} \qquad \text{and} \qquad \|u_i\|_{L^\infty(\partial B_1)} < \varepsilon_\beta$$

with ε_β as in Theorem 6.3.

Step three: We conclude by deriving a contradiction by applying Theorem 6.3. To this end set $G_i = x_i + t_i F_i$ where x_i is such that $\int_{G_i} x\,\mathrm{d}x = 0$ and $t_i > 0$ is such that $|G_i| = |B_1|$. Up to increasing the value of Λ, we find $v_i \in C^{1,\gamma}(\partial B_1)$ such that

$$\partial G_i = \Big\{(1+v_i(x))x \,:\, x \in \partial B_1\Big\} \qquad \text{and} \qquad \|v_i\|_{L^\infty(\partial B_1)} < \varepsilon_\beta.$$

We can now apply Theorem 6.3 to each G_i to conclude

$$\mathcal{E}_{\alpha,\beta}(B_1) + C\left(1 - \frac{\beta}{\beta_*(\alpha)}\right)\|v_i\|^2_{H^1(\partial B_1)} \leqslant \mathcal{E}_{\alpha,\beta}(G_i) = t_i^2\mathcal{P}(F_i) + \beta\, t_i^5\mathcal{N}_{t_i\alpha}(F_i). \tag{6.22}$$

We first show that $t_i \geqslant 1$. To this end we prove when $t_i < 1$ that

$$t_i^3\mathcal{N}_{t_i\alpha}(F_i) \leqslant \mathcal{N}_\alpha(F_i) + c|F_i\Delta E_i| \tag{6.23}$$

for some $c > 0$ depending only on α. By the Lipschitz estimate (6.20) we have

$$|\mathcal{N}_\alpha(E_i) - \mathcal{N}_\alpha(F_i)| \leqslant \frac{8\pi^2}{\alpha^2}|F_i\Delta E_i|,$$

so that, since $t_i < 1$,

$$t_i^3\mathcal{N}_{t_i\alpha}(F_i) \leqslant \mathcal{N}_\alpha(F_i) + \frac{16\pi^2}{\alpha^2}|F_i\Delta E_i| + |t_i^3\mathcal{N}_{t_i\alpha}(E_i) - \mathcal{N}_\alpha(E_i)|.$$

We estimate the latter difference as

$$|t_i^3\mathcal{N}_{t_i\alpha}(E_i) - \mathcal{N}_\alpha(E_i)| \leqslant (1 - t_i^3)\,\mathcal{N}_{t_i\alpha}(E_i) + |\mathcal{N}_{t_i\alpha}(E_i) - \mathcal{N}_\alpha(E_i)|.$$

For the first term, since $|E_i| = |B_1|$, then by Lemma 2.1(iv)

$$(1 - t_i^3)\,\mathcal{N}_{t_i\alpha}(E_i) \leqslant 3(1 - t_i)\mathcal{N}_0(B_1) \leqslant \frac{32\pi^2}{5}(1 - t_i).$$

For the second term, using again $|E_i| = |B_1|$, we have

$$|\mathcal{N}_{t_i\alpha}(E_i) - \mathcal{N}_\alpha(E_i)| \leqslant \int_{E_i}\int_{E_i}\frac{|e^{-t_i\alpha|x-y|} - e^{-\alpha|x-y|}|}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \leqslant \alpha(1 - t_i)|B_1|^2.$$

Finally, since $|t_iF_i| = |B_1|$ and $t_i < 1$ then

$$|F_i\Delta E_i| \geqslant ||F_i| - |B_1|| = |B_1|\left(\frac{1}{t_i^3} - 1\right) \geqslant |B_1|(1 - t_i),$$

and (6.23) is proved with $c = 16\pi^2/\alpha^2 + 24\pi/5 + 4\pi\alpha/3$. Finally, from (6.21) and (6.23) we have

$$\begin{aligned} t_i^2\mathcal{P}(F_i) + \beta\, t_i^5\mathcal{N}_{t_i\alpha}(F_i) &\leqslant t_i^2\left(\mathcal{E}_{\alpha,\beta}(F_i) + c\beta|F_i\Delta E_i|\right) \\ &\leqslant t_i^2\left(\mathcal{E}_{\alpha,\beta}(B_1) + (c\beta - \Lambda)|F_i\Delta E_i|\right) < \mathcal{E}_{\alpha,\beta}(B_1) \end{aligned}$$

in contradiction to (6.22) if $\Lambda > c\beta$, which depends only on α and β. Hence $t_i \geqslant 1$ as desired. In this case, we have $\mathcal{N}_{t_i\alpha}(F_i) \leqslant \mathcal{N}_\alpha(F_i)$, and applying (6.21) in (6.22) gives

$$t_i^{-5}\mathcal{E}_{\alpha,\beta}(B_1) + Ct_i^{-5}(1 - \beta/\beta_*(\alpha))\|v_i\|^2_{H^1(\partial B_1)} + \Lambda|F_i\Delta E_i| \leqslant \mathcal{E}_{\alpha,\beta}(B_1). \tag{6.24}$$

Now, if $t_i = 1$ then $F_i = E_i$ and $\|v_i\|_{H^1(\partial B_1)} = 0$. Hence $E_i = B_1(-x_i)$, a contradiction to (6.16) unless equality holds. If instead $t_i > 1$ then $|F_i\Delta E_i| \geqslant ||F_i| - |B_1|| = |B_1|(1 - t_i^{-3})$. Applying this in (6.24) gives

$$\Lambda|B_1|\left(1 - \frac{1}{t_i^3}\right) \leqslant \mathcal{E}_{\alpha,\beta}(B_1)\left(1 - \frac{1}{t_i^5}\right)$$

where for instance taking $\Lambda > 2\mathcal{E}_{\alpha,\beta}(B_1)$ yields a contradiction. □

Remark 6.4. We can deduce the rigidity statement in Theorem 6.1 by keeping the $\|v_i\|_{H^1(\partial B_1)}$ term in (6.24). In [FFM+15, Lemma 8.5], the authors choose to discard this term; were they to keep it, they would also deduce the same rigidity.

Appendix A. Characterization of the critical threshold $\beta_*(\alpha)$

Here we prove the identity (5.11) by obtaining the monotonicity of the sequence that defines $\beta_*(\alpha)$ as in (5.10). Similar properties of eigenvalues have been shown in [BCT24, FFM+15], where in the Riesz setting the analogous eigenvalues are ratios of Gamma functions, which exhibit a product recursion across modes and the infimum of ratios can be identified via an induction argument. Such a strategy is not available here. Instead, following the classical literature on potential theory and boundary integral equations (cf. [HW08, McL00]), we use a variational characterization of the eigenvalues, which replaces the product structure. To this end, we define

$$\beta_*^{(l)}(\alpha) \coloneqq \frac{\lambda_l - \lambda_1}{\mu_l^\alpha - \mu_1^\alpha}.$$

Our goal is to prove the following result.

Proposition A.1. *For any $\alpha > 0$ the sequence $(\beta_*^{(l)}(\alpha))_{l\in\mathbb{N}}$ is strictly increasing.*

Let us define the functionals

$$A(\varphi) \coloneqq \int_0^\infty \Big((\varphi'(r))^2 + \alpha^2\varphi^2(r)\Big)\, r^2\,\mathrm{d}r, \qquad B(\varphi) \coloneqq \int_0^\infty \varphi^2(r)\,\mathrm{d}r,$$

over the class $\mathscr{A} \coloneqq \big\{\varphi \in AC_{\mathrm{loc}}((0,\infty))\colon A(\varphi) + B(\varphi) < \infty\big\}$. For any $\tau \geqslant 0$, let

$$G(\tau) \coloneqq \inf\Big\{A(\varphi) + \tau B(\varphi)\colon \varphi \in \mathscr{A},\ \varphi(1) = 1\Big\} \tag{A.1}$$

such that $G(\lambda_l)$ is the minimal screened Dirichlet energy among extensions of Y_l^m to $\mathbb{R}^3$ with trace Y_l^m on the unit sphere.

Lemma A.2. *For any $l \geqslant 1$, $G(\lambda_l) = \dfrac{1}{\alpha\, i_l(\alpha) k_l(\alpha)}$, and the infimum in (A.1) at $\tau = \lambda_l$ is attained at*

$$\varphi_l(r) \coloneqq \begin{cases} \dfrac{i_l(\alpha r)}{i_l(\alpha)} & \text{if } 0 < r \leqslant 1, \\[2mm] \dfrac{k_l(\alpha r)}{k_l(\alpha)} & \text{if } r \geqslant 1. \end{cases}$$

Proof. Using the behavior of i_l at $r=0$ and k_l at $r=\infty$ we see that $\varphi_l \in \mathscr{A}$. Also, φ_l is continuous at $r = 1$ and $\varphi_l(1) = 1$. Since i_l and k_l solve the modified spherical Bessel equation $x^2u'' + 2xu' - \big(x^2 + l(l+1)\big)u = 0$, φ_l solves $\big(r^2\varphi'\big)' = \big(\lambda_l + \alpha^2 r^2\big)\varphi$ on $(0,1)$ and $(1,\infty)$, which is exactly the Euler–Lagrange equation of $Q(\varphi) \coloneqq A(\varphi) + \lambda_l B(\varphi)$.

Now let $\varphi \in \mathscr{A}$ be arbitrary with $\varphi(1) = 1$. Define $w \coloneqq \varphi - \varphi_l$. Then $w \in \mathscr{A}$ with $w(1) = 0$, and $Q(\varphi) = Q(\varphi_l) + 2I(\varphi_l, w) + Q(w)$, where

$$I(\varphi_l, w) \coloneqq \int_0^\infty \big(r^2\varphi_l' w' + (\lambda_l + \alpha^2 r^2)\varphi_l w\big)\,\mathrm{d}r.$$

We claim that $I \equiv 0$. To prove this, first note that since φ_l solves the Euler–Lagrange equation on $(0,1)$ and $(1,\infty)$, we have

$$I(\varphi_l, w) = \int_0^1 \big(r^2\varphi_l' w\big)'\,\mathrm{d}r + \int_1^\infty \big(r^2\varphi_l' w\big)'\,\mathrm{d}r.$$

Hence, for $0 < \varepsilon < 1 < R$, we get

$$\int_{\varepsilon}^{R} \left(r^2 \varphi_l' w\right)' \, \mathrm{d}r = R^2 \varphi_l'(R) w(R) - \varepsilon^2 \varphi_l'(\varepsilon) w(\varepsilon),$$

since $w(1) = 0$.

For $0 < \varepsilon < 1/2$ we estimate

$$|w(\varepsilon)| \leqslant |w(1/2)| + \int_{\varepsilon}^{1/2} |w'| \, \mathrm{d}r \leqslant |w(1/2)| + A^{1/2}(w) \left(\int_{\varepsilon}^{1/2} \frac{\mathrm{d}r}{r^2} \right)^{1/2} \leqslant C_w \varepsilon^{-1/2}.$$

On the other hand, using the properties of i_l, $\varepsilon^2 \varphi_l'(\varepsilon) = O(\varepsilon^{l+1})$. Thus $\varepsilon^2 \varphi_l'(\varepsilon) w(\varepsilon) = O(\varepsilon^{l+1/2}) \to 0$ for $l \geqslant 1$ as $\varepsilon \to 0$.

Since $B(w) < \infty$ we have that $w \in L^2((0,\infty))$. So, there exist $R_j \in [j, 2j]$ with $w^2(R_j) \leqslant \frac{1}{j} \int_j^{2j} w^2 \, \mathrm{d}r \to 0$. Again, a direct calculation using the properties of k_l implies that $R^2 \varphi_l'(R) \to 0$ as $R \to \infty$. Thus $R^2 \varphi_l'(R) w(R) \to 0$ as $R \to \infty$. Then the claim follows sending $\varepsilon \to 0$ and $R \to \infty$.

As a consequence we obtain that $Q(\varphi) = Q(\varphi_l) + Q(w)$. Since $Q(w) \geqslant \alpha^2 \int_0^\infty w^2 r^2 \, \mathrm{d}r \geqslant 0$, and $Q(w)$ vanishes only when $w = 0$ we conclude that φ_l is the unique minimizer of Q.

Therefore, taking φ_l instead of w,

$$G(\lambda_l) = Q(\varphi_l) = \varphi_l'(1^-) - \varphi_l'(1^+) = \alpha \left(\frac{i_l'(\alpha)}{i_l(\alpha)} - \frac{k_l'(\alpha)}{k_l(\alpha)} \right) = \frac{1}{\alpha \, i_l(\alpha) k_l(\alpha)},$$

where we used the fact that

$$i_l(\alpha) k_l'(\alpha) - i_l'(\alpha) k_l(\alpha) = \frac{1}{\alpha} \left(I_{l+1/2}(\alpha) K_{l+1/2}'(\alpha) - I_{l+1/2}'(\alpha) K_{l+1/2}(\alpha) \right) = -\frac{1}{\alpha^2}$$

by direct computation. □

The next lemma establishes the properties of the function G; its monotonicity and concavity are key in proving the monotonicity of the eigenvalues.

Lemma A.3. *The function G is finite, nonnegative and concave on $[0, \infty)$. Moreover we have $G(\lambda_j) < G(\lambda_l)$ whenever $1 \leqslant j < l$.*

Proof. Nonnegativity of G clearly follows by its definition. It is also finite since, for example, $A(\varphi_1) + \tau B(\varphi_1) < \infty$. Furthermore the map $\tau \mapsto A(\varphi) + \tau B(\varphi)$ is affine with positive slope. Thus, being a positive infimum of increasing affine functions over a τ-independent index, the function G is increasing and concave. Now let φ_l be the minimizer in Lemma A.2 above. Then

$$G(\lambda_j) \leqslant A(\varphi_l) + \lambda_j B(\varphi_l) = G(\lambda_l) - (\lambda_l - \lambda_j) B(\varphi_l) < G(\lambda_l),$$

as claimed. □

We now prove Proposition A.1.

Proof of Proposition A.1. Note that by Lemma A.2,

$$\mu_l^\alpha - \mu_1^\alpha = 8\pi\left(\frac{1}{G(\lambda_1)} - \frac{1}{G(\lambda_l)}\right);$$

hence, it suffices to prove that

$$\frac{\lambda_j - \lambda_1}{1/G(\lambda_1) - 1/G(\lambda_j)} < \frac{\lambda_l - \lambda_1}{1/G(\lambda_1) - 1/G(\lambda_l)}. \tag{A.2}$$

Let $t := (\lambda_l - \lambda_j)/(\lambda_l - \lambda_1) \in (0,1)$ so that $\lambda_j = t\lambda_1 + (1-t)\lambda_l$. Since G is concave,

$$\frac{1}{G(\lambda_j)} \leqslant \frac{1}{tG(\lambda_1) + (1-t)G(\lambda_l)}.$$

Now, note that

$$\big(tG(\lambda_1) + (1-t)G(\lambda_l)\big)\left(\frac{t}{G(\lambda_1)} + \frac{1-t}{G(\lambda_l)}\right)$$
$$= t^2 + (1-t)^2 + t(1-t)\left(\frac{G(\lambda_1)}{G(\lambda_l)} + \frac{G(\lambda_l)}{G(\lambda_1)}\right) > 1,$$

where we used the fact that $x + x^{-1} > 2$ for $x \neq 1$. Thus

$$\frac{1}{G(\lambda_j)} < \frac{t}{G(\lambda_1)} + \frac{1-t}{G(\lambda_l)},$$

which, in turn, implies that

$$\frac{1}{G(\lambda_1)} - \frac{1}{G(\lambda_j)} > (1-t)\left(\frac{1}{G(\lambda_1)} - \frac{1}{G(\lambda_l)}\right) = \frac{\lambda_j - \lambda_1}{\lambda_l - \lambda_1}\left(\frac{1}{G(\lambda_1)} - \frac{1}{G(\lambda_l)}\right).$$

Since all four differences are positive (A.2) follows. □

References

[AFM13] Emilio Acerbi, Nicola Fusco, and Massimiliano Morini. Minimality via second variation for a nonlocal isoperimetric problem. *Comm. Math. Phys.*, 322(2):515–557, 2013.

[AJ76] F. J. Almgren Jr. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.*, 4(65):viii+199, 1976.

[AWH13] George B. Arfken, Hans J. Weber, and Frank E. Harris. *Mathematical Methods for Physicists (Seventh Edition).* Academic Press, Boston, 2013.

[BC14] Marco Bonacini and Riccardo Cristoferi. Local and global minimality results for a nonlocal isoperimetric problem on $\mathbb{R}^N$. *SIAM J. Math. Anal.*, 46(4):2310–2349, 2014.

[BCT24] Marco Bonacini, Riccardo Cristoferi, and Ihsan Topaloglu. Stability of the ball for attractive-repulsive energies. *SIAM J. Math. Anal.*, 56:588–615, 2024.

[BMP25] Lia Bronsard, Benoît Merlet, and Marc Pegon. Non-spherical minimizers in the generalized liquid drop model for Yukawa and truncated Coulomb potentials, 2025.

[CFP23] Davide Carazzato, Nicola Fusco, and Aldo Pratelli. Minimality of balls in the small volume regime for a general Gamow-type functional. *Adv. Calc. Var.*, 16(2):503–515, 2023.

[CG26] Otis Chodosh and Matilde Gianocca. No compromise in the liquid drop model, 2026.

[CH10] Andrey I. Chugunov and Charles J. Horowitz. Breaking stress of neutron star crust. *Mon. Not. R. Astron. Soc. Lett.*, 407:L54, 2010.

[CH17] Matthew E. Caplan and Charles J. Horowitz. Colloquium: Astromaterial science and nuclear pasta. *Rev. Mod. Phys.*, 89(4):041002, 2017.

[CMT17] Rustum Choksi, Cyrill B. Muratov, and Ihsan Topaloglu. An old problem resurfaces nonlocally: Gamow's liquid drops inspire today's research and applications. *Notices Amer. Math. Soc.*, 64(11):1275–1283, 2017.

[FFM+15] Alessio Figalli, Nicola Fusco, Francesco Maggi, Vincent Millot, and Massimiliano Morini. Isoperimetry and stability properties of balls with respect to nonlocal energies. *Comm. Math. Phys.*, 336(1):441–507, 2015.

[FL15] Rupert L. Frank and Elliott H. Lieb. A compactness lemma and its application to the existence of minimizers for the liquid drop model. *SIAM J. Math. Anal.*, 47(6):4436–4450, 2015.

[FN21] Rupert L. Frank and Phan Thành Nam. Existence and nonexistence in the liquid drop model. *Calc. Var. Partial Differential Equations*, 60(6):Paper No. 223, 2021.

[Fug89] Bent Fuglede. Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbb{R}^n$. *Trans. Amer. Math. Soc.*, 314(2):619–638, 1989.

[GMP24] Michael Goldman, Benoît Merlet, and Marc Pegon. Uniform $C^{1,\alpha}$-regularity for almost-minimizers of some nonlocal perturbations of the perimeter. *Arch. Ration. Mech. Anal.*, 248:102, 2024.

[HSY84] Masa-aki Hashimoto, Hironori Seki, and Masami Yamada. Shape of nuclei in the crust of a neutron star. *Prog. Theor. Phys.*, 71(2):320–326, 1984.

[HW08] George C. Hsiao and Wolfgang L. Wendland. *Boundary Integral Equations*. Springer, Berlin, 2008.

[Isr11] Jacob N. Israelachvili. *Intermolecular and Surface Forces*. Academic Press, Amsterdam, 3 edition, 2011.

[LL01] Elliott H. Lieb and Michael Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2 edition, 2001.

[Mag12] Francesco Maggi. *Sets of Finite Perimeter and Geometric Variational Problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012.

[McL00] William McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, Cambridge, 2000.

[MNS25] Cyrill B. Muratov, Matteo Novaga, and Thilo M. Simon. Euler's elastica functional as a large mass limit of a two-dimensional non-local isoperimetric problem, 2025.

[NP21] Matteo Novaga and Aldo Pratelli. Minimisers of a general Riesz-type problem. *Nonlinear Anal.*, 209:112346, 2021.

[Peg21] Marc Pegon. Large mass minimizers for isoperimetric problems with integrable nonlocal potentials. *Nonlinear Anal.*, 211:112395, 2021.

[PR95] Christopher J. Pethick and David G. Ravenhall. Matter at large neutron excess and the physics of neutron-star crusts. *Annu. Rev. Nucl. Part. Sci.*, 45:429–484, 1995.

[RPW83] David G. Ravenhall, Christopher J. Pethick, and James R. Wilson. Structure of matter below nuclear saturation density. *Phys. Rev. Lett.*, 50(26):2066–2069, 1983.

[Seg11] Javier Segura. Bounds for ratios of modified Bessel functions and associated Turán-type inequalities. *J. Math. Anal. Appl.*, 374(2):516–528, 2011.

[VO48] Evert Johannes Willem Verwey and Jan Theodoor Gerard Overbeek. *Theory of the Stability of Lyophobic Colloids*. Elsevier, Amsterdam, 1948.

[WDNWB21] Miriam Winkelmann, Edoardo Di Napoli, Daniel Wortmann, and Stefan Blügel. Solution to the modified Helmholtz equation for arbitrary periodic charge densities. *Front. Phys.*, 8:618142, 2021.

(Lia Bronsard) Department of Mathematics & Statistics, McMaster University, Hamilton ON, Canada

Email address: bronsard@mcmaster.edu

(Kenneth DeMason) Department of Mathematics & Statistics, McMaster University, Hamilton ON, Canada

Email address: demasonk@mcmaster.edu

(Ihsan Topaloglu) Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA, USA

Email address: iatopaloglu@vcu.edu